The Mathematics of Zome

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Abstract

We will calculate the lengths of all the Zome struts and show that the locations of the Zome balls in an arbitrary configuration satisfy a very simple rule that involves the golden ratio.

1 Introduction

The Zome system is a construction system based on a set of plastic struts and balls that can be attached together to form an amazing set of mathematically or artistically interesting structures. There is a fair amount of deep mathematics involved, and the purpose of this article is to look at some of that.

Anyone who has played with the Zome system is usually amazed at first how well things work out. It allows one to construct a huge number of structures, but no matter how complex the structure, it seems that if two zome balls are near each other and each has a hole pointing at the other, the holes will be of the correct shape and in the correct orientation that a standard Zome strut will connect them. There is a mathematical reason for this, and the purpose of this article is to demonstrate that reason. The article does not concern how to build complex Zome structures. For that sort of information, there are hundreds of resources on the internet.

It will be much easier to follow along if you have at least a small set of Zome parts (and the larger your set, the better, although large sets tend to become expensive). It is also useful for you to have fooled around a bit with the system before trying to understand some of the mathematics expressed here. This article includes photos of various Zome structures, but you will usually find the arguments much easier to follow if you build a physical model of each of those structures yourself so that you can more easily manipulate the 3-D version and see it from many points of view.

For information on Zome and for an on-line way to order kits or parts, see:

`http://www.zometool.com`

The main Zome strut colors are red, yellow and blue and most of what we'll cover here will use those as examples. There are green struts that are necessary for building structures with regular tetrahedrons and octahedrons, and almost everything we say about the red, yellow and blue struts will apply to the green ones. The green ones are a little harder to work with (both physically and mathematically) because they have a pentagon-shaped head, but can fit into any pentagonal hole in five different orientations. With the regular red, yellow and blue struts there is only one way to insert a strut into a Zome ball hole. The blue-green struts are not really part of the Zome geometry (because they have the "wrong" length). They are necessary for building a few of the Archimedian solids, like the rhombicuboctahedron and the truncated cuboctahedron.

2 The Zome Ball

Look carefully at a Zome ball. (It is better to look at a physical ball, but an image of one appears in Figure 1. It is highly symmetric, and has holes that will accept struts of three shapes: rectangles with an aspect ratio of roughly $1:2$, equilateral triangles and regular pentagons. Every pentagonal hole looks the same: it is surrounded by 5 rectangular holes and 5 triangular holes. The same can be said of every hole: the shapes and orientations of the neighboring holes are the same for every hole in the ball.

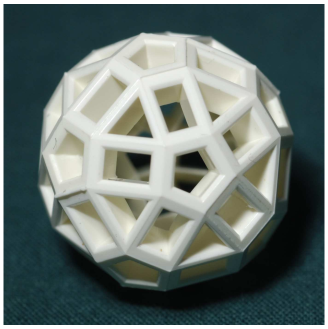

Figure 1: The Zome Ball

Another way to convince yourself that all the holes of a certain shape are basically identical is to place a Zome ball on a table balanced on a hole of a particular shape (say a rectangle). Now take another Zome ball and place it on *any* of its rectangular holes (or hole of the same shape as the first ball). If you rotate the second ball so that the rectangles on top have the same orientation, you will find that every hole matches in shape and direction in the two balls.

There are 12 pentagon-shaped holes and if you imagine that the pentagons were all left in their planes but expanded until their edges touched the nearest pentagon edges, the resulting figure would be a dodecahedron (a regular 12-sided polyhedron).

If you think about this pentagon expansion, every pair of adjacent pentagons would close over a rectangular hole, so there are the same number of rectangular holes as there are edges in a dodecahedron; namely, 30.

Finally, again visualizing the expansion of the pentagonal holes, each triangle on the Zome ball will be covered by a vertex of the final dodecahedron, so there are the same number of triangles as there are vertices of a dodecahedron; namely, 20.

A dual argument can be made: instead of expanding the pentagons until their edges merge, expand the triangles in the same way, and the resulting figure will be a regular icosahedron – a polyhedron with 20 identical triangluar sides. Each vertex of the resulting icosahedron (of which there are 12) corresponds to a pentagonal hole in the Zome ball and each edge of the icosahedron (of which there are 30) corresponds to one of the rectangular holes in the Zome ball.

Luckily, we obtain the same counts using both approaches: 12 pentagonal holes, 20 triangular holes and 30 rectangular holes for a total of $12 + 20 + 30 = 62$ holes.

Finally, let's look at the orientations of Zome balls in any sort of structure. If you hook together any sort of combinations of balls and struts in such a way that none of the struts are forced to bend, then every Zome ball in the entire structure will have exactly the same orientation: if one ball has a rectangle pointing in a particular direction with a particular orientation, then *every* ball in the structure will have one of its rectangles pointing in the same direction and with the same orientation. Figure 2 shows a central ball with one strut of each of the major colors and with a Zome ball attached to the end of each. Note that the balls all have the same orientation.

It is pretty easy to convince yourself of this: just place a strut of each shape into a Zome ball and look at the balls on the other ends of the struts. In every case, the adjacent ball has the same orientation, so if you begin at any ball and follow a sequence of struts to another one, you'll wind up at a ball that has the same orientation. This is true even of the green (and even of the blue-green) struts.

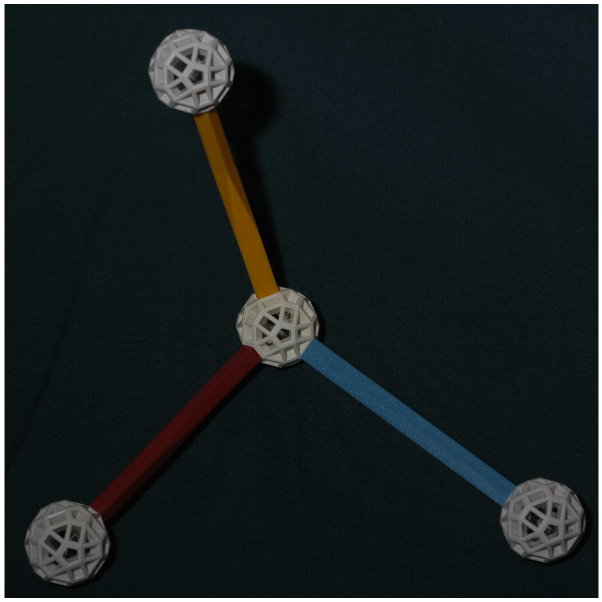

Figure 2: Zome Ball Orientation

3 Zome Struts

In a standard Zome set there are struts of three lengths in each color: red, yellow and blue. It is now possible to purchase other lengths of some of them: short and super-short. We will call the original nine struts the "standard struts".

The usual labeling of the standard struts is B_1, B_2, B_3, Y_1, Y_2, Y_3, R_1, R_2 and R_3 for the blue, yellow and red struts, and the smaller numbers refer to the shorter struts. Thus B_1 is the shortest blue strut and R_3 is the longest red one.

The physical lengths of the struts are chosen so that their mathematical lengths are perfect. If we consider a structure that has a Zome ball at the end of every strut, in a mathematical sense, the centers of the Zome balls should be considered to be the endpoints of the struts embedded in the balls. Thus the "true" mathematical length of a strut should be the distance between the centers of two Zome balls attached to the ends of the strut.

Measuring in terms of these mathematical lengths, the strut lengths follow a very regular pattern: in each color, the lengths increase by a constant factor called τ (this is the Greek letter "tau" and it stands for the so-called "golden ratio": a number that is approximately 1.618).

Thus the mathematical length of B_2 is τ times as long as a B_1, and the length of B_3 is τ times the length of B_2 (or τ^2 times the length of B_1). The same relation holds for struts of every color, including the greens and even the blue-greens.

We will assign an arbitrary length of 1 to B_1, the shortest blue strut. With this assignment, the lengths of B_2 and B_3 are therefore τ and τ^2, respectively.

From Figure 3 we can glean enough information to calculate the lengths of the yellow struts. The planar figure is made of two perpendicular blue struts, B_1 and B_3. The three yellow struts all are of type Y_2. It is clear from the construction that a yellow Y_2 strut is the hypotenuse of a right triangle whose sides have lengths $1/2$ and $\tau^2/2$. Using the Pythagorean theorem and a bit of messy algebra (which we will be able to do much more easily after we have examined some properties of τ in Section 4.1) yields the result that the length of the Y_2 yellow strut is $\tau(\sqrt{3}/2)$. This implies that the length of Y_1 is $(\sqrt{3}/2)$ and the length of Y_3 is $\tau^2(\sqrt{3}/2)$ since all the struts of the same color have lengths that are simply multiples of τ of each other.

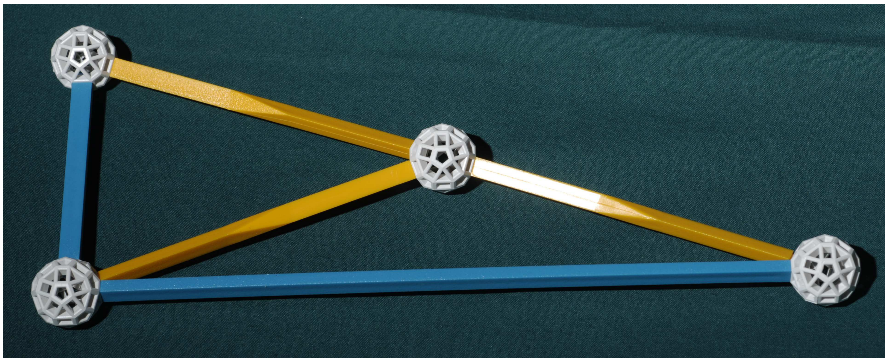

Figure 3: Yellow Struts

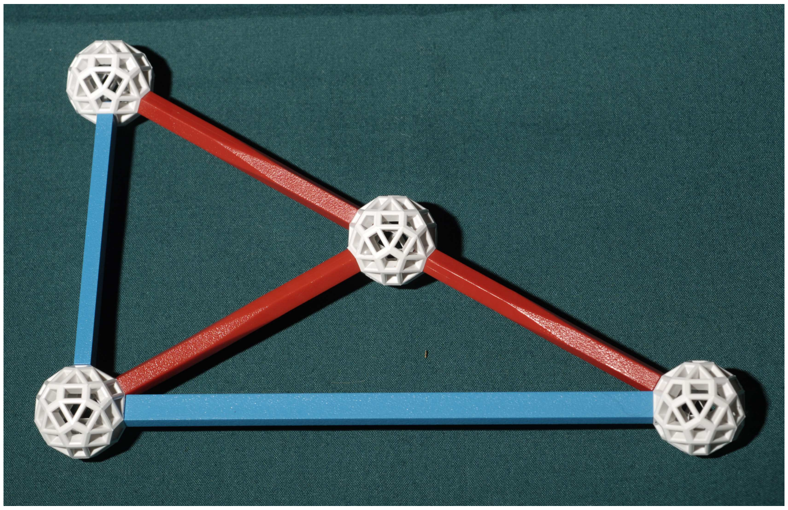

Figure 4: Red Struts

Similarly, the length of the red struts can be calculated from the planar Figure 4. This time the structure is built from two perpendicular blue struts B_1 and B_2 and the red struts are all R_1. This time the R_1 forms the hypotenuse of a right triangle whose other sides have lengths $1/2$ and $\tau/2$. Again, an ugly calculation yields the length of R_1 as $\sqrt{2+\tau}/2$, so the lengths of R_2 and R_3 are $\tau\sqrt{2+\tau}/2$ and $\tau^2\sqrt{2+\tau}/2$, respectively. See Section 4.1 for the details of the calculation.

4 The Golden Ratio τ

The number τ is defined to be the largest root of the equation

$$x^2 = x + 1. \tag{1}$$

which can easily be solved using the quadratic formula or any other method, and the resulting value for τ is $(1+\sqrt{5})/2$. But equation 1 allows us to express powers of τ in terms of τ itself. Obviously we obtain directly from equation 1 that $\tau^2 = \tau + 1$.

What about τ^3? Well, $\tau^3 = \tau(\tau^2)$ and we know that $\tau^2 = \tau + 1$, so $\tau^3 = \tau(\tau+1) = \tau^2 + \tau = 2\tau + 1$.

We can, using the same approach, obtain expressions for τ^4, τ^5, and so on, yielding the following table:

$$
\begin{aligned}
\tau^1 &= 1\tau + 0 \\
\tau^2 &= 1\tau + 1 \\
\tau^3 &= 2\tau + 1 \\
\tau^4 &= 3\tau + 2 \\
\tau^5 &= 5\tau + 3 \\
\tau^6 &= 8\tau + 5
\end{aligned}
$$

A quick glance at the table shows that the coefficients of the expansion are just the Fibonacci numbers, and a simple inductive argument shows us that this is true.

Since the lengths of the Zome struts of any color are just multiples of a basic length by some power of τ, then every strut length can also be expressed as an integer linear combination of 1 and τ times the length of the basic strut of that color.

A standard Zome set has a certain shortest length of the blue strut which we defined to be 1, but in principle, shorter struts could be obtained by continuing to divide the lengths by τ. It is somewhat amazing, but dividing by powers of τ also yields integer combinations of 1 and τ. For example,

$$
\frac{1}{\tau} = \tau - 1,
$$

and this can be obtained from equation 1 by dividing both sides by τ and regrouping terms. But now that the value of $1/\tau$ is known, we can obtain in a similar way values for $1/\tau^2$, $1/\tau^3$, et cetera, and the table above can be extended in the negative direction. It is a good exercise for the reader to check these values:

$$
\begin{aligned}
\tau^{-4} &= -3\tau + 5 \\
\tau^{-3} &= 2\tau - 3 \\
\tau^{-2} &= -1\tau + 2 \\
\tau^{-1} &= 1\tau - 1 \\
\tau^{0} &= 0\tau + 1 \\
\tau^{1} &= 1\tau + 0 \\
\tau^{2} &= 1\tau + 1 \\
\tau^{3} &= 2\tau + 1 \\
\tau^{4} &= 3\tau + 2
\end{aligned}
$$

The coefficients for the negative powers of τ look a little strange: the signs alternate, but the values are just the familiar Fibonacci numbers. The usual definiton of the Fibonacci numbers starts with $F_0 = 0$ and $F_1 = 1$, and once any two adjacent values are known, the next is obtained using the recurrence relation $F_n = F_{n-1} + F_{n-2}$.

The nice thing is that the same recurrence relation can be used to obtain "reasonable" values for F_{-1}, F_{-2} and so on. What should be the value, for example, of F_{-1}? Well, it should satisfy:

$$
F_{-1} + F_0 = F_1,
$$

and since we know the values of F_0 and F_1, we obtain $F_{-1} = 1$.

Continuing in the same way, we can obtain $F_{-2} = -1$, $F_{-3} = 2$, $F_{-4} = -3$ and so on. Note that these follow the same pattern as the coefficients of the simplified negative powers of τ in the table above. It is easy to show that this pattern continues. In fact, the general formula for τ^n, where n is positive, negative, or zero, is given by:

$$
\tau^n = F_n\tau + F_{n+1},
$$

where the values of the Fibonacci numbers are extended to negative values as described above.

4.1 Calculating Strut Lengths

Using the relations we have discovered in the first part of this section, it is easy to work out the lengths of the red and yellow struts in terms of the blue ones.

Let's begin with the yellow struts. As we saw in Section 2, the length of Y_2 is the hypotenuse of a right triangle with sides having lengths $1/2$ and $\tau^2/2$. If that unkown length is y_2, we have:

$$y_2 = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\tau^2}{2}\right)^2}.$$

Thus:

$$y_2^2 = \frac{1+\tau^4}{4}.$$

Since $\tau^2 = 1 + \tau$, we can square both sides to obtain $\tau^4 = (1+\tau)^2$ which we can substitute into the equation above:

$$\begin{aligned} y_2^2 &= \frac{1+(1+\tau)^2}{4} \\ &= \frac{1+1+2\tau+\tau^2}{4} \\ &= \frac{2(1+\tau)+\tau^2}{4}. \end{aligned}$$

Now, since $(1+\tau) = \tau^2$ we can substitute τ^2 for $(1+\tau)$ in the equation above and obtain:

$$\begin{aligned} y_2^2 &= \frac{2\tau^2+\tau^2}{4} \\ y_2^2 &= \frac{3\tau^2}{4} \\ y_2 &= \tau\frac{\sqrt{3}}{2}, \end{aligned}$$

which leads to the values for the lengths of Y_1, Y_2 and Y_3: $\sqrt{3}/2$, $\tau\sqrt{3}/2$, and $\tau^2\sqrt{3}2$, respectively.

To calculate the length of the red struts, we again use the Pythagorean theorem in conjunction with the structure illustrated in Figure 4. The length r_1 of R_1 is given by:

$$\begin{aligned} r_1 &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\tau}{2}\right)^2} \\ r_1^2 &= \frac{1}{4} + \frac{\tau^2}{4} \\ &= \frac{1+\tau^2}{4}. \end{aligned}$$

Since $\tau^2 = 1 + \tau$, this becomes:

$$\begin{aligned} r_1^2 &= \frac{2+\tau}{4} \\ r_1 &= \frac{\sqrt{2+\tau}}{2}. \end{aligned}$$

Thus the lengths of R_2 and R_3 are $\tau\sqrt{2+\tau}/2$ and $\tau^2\sqrt{2+\tau}/2$, respectively.

5 Zome Ball Coordinates

Because of the angles of the Zome ball holes, it turns out that any combination of red, blue, yellow (and even green) struts will leave a ball at a point with nice coordinates, based on the following coordinate system:

As before, call the length of the shortest blue strut 1. Plug three of these shortest blues into a Zome ball so that they are all mutually perpendicular, and form a right-handed coordinate system. If that original Zome ball has coordinates $(0, 0, 0)$, then the centers of Zome balls stuck on the ends of each of the coordinate axis struts will have coordinates $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. This will be our coordinate system and we will now examine the coordinates of the centers of Zome balls which are hooked together in an arbitrary fashion to a ball that is designated to lie at the origin of our system with some set of short blues marking the x, y and z axes.

It turns out that no matter what combinations of struts of any lengths are stuffed into the holes, with any linkage whatsoever, the coordinates of any reachable Zome ball will have the form (α, β, γ), where α, β and γ are numbers of the form $(a\tau + b)/2$, where a and b are integers (possibly negative or zero) and τ is the golden ratio: $\tau = (1 + \sqrt{5})/2 \approx 1.618039887$.

Now we will demonstrate our main result, that the coordinates of any zome ball reachable from a Zome ball at $(0, 0, 0)$ all have the form:

$$(\frac{a\tau + b}{2}, \frac{c\tau + d}{2}, \frac{e\tau + f}{2}) \qquad (2)$$

What we need to do to prove this is to show that when any strut is plugged into any Zome ball hole having coordinates like those above, the coordinates of a ball placed at the end of the strut will also have coordinates with the same property. Since the ball at the origin satisfies the condition (namely $a = b = c = d = e = f = 0$) then the condition is preserved as each strut is added to any spatial path.

Note: It is possible to purchase special “half blue” and “half green” struts that are half the length of the normal ones. If these are allowed in the system, the result holds except that the denominator in Equation 2 must be changed from 2 to 4.

This is true of the red, blue, yellow and green struts, but we will only prove it here for the red, blue and yellow ones. The proof for the green struts is similar, but there are five possibilities to consider since a green strut can be pushed into a pentagonal hole in five different orientations. Working out the details for the green struts is not difficult, but there are five cases, so it is a good exercise to do this.

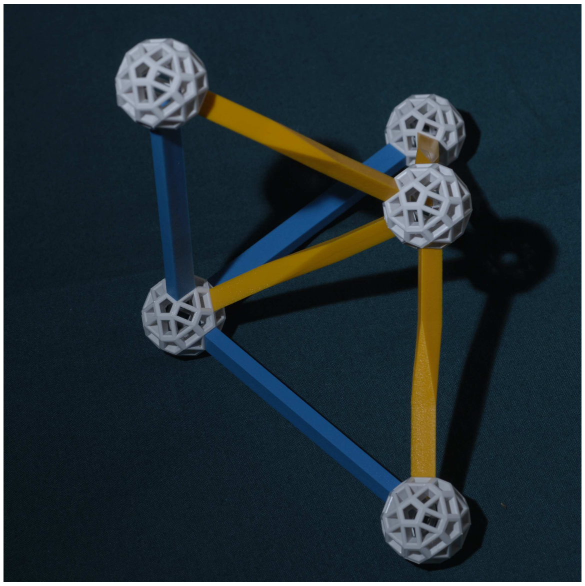

Figure 5: Yellow Strut

We will demonstrate our result for the red, blue and yellow struts by showing that if a particular strut of a particular color is put in the Zome ball at the origin, the coordinates of the center of the ball at the end of the strut have the correct form. Then, since the actual strut used can have a length that is a multiple by a power of τ of the particular strut, the coordinates for a different length strut will simply be multiplied by some power of τ. But we have shown that such a multiplication will just yield another integer combination of 1 and τ, so every strut coming out of the origin will yield an endpoint having the form shown in equation 2.

Finally, since we will be, in general, placing the strut into a ball that is not at the origin, but into a ball whose coordinates have the form shown in equation 2, we will simply be adding coordinates of those forms, and the resulting ball at the end of the strut will have the required coordinates.

Insert three short blue struts to represent the perpendicular coordinate axes and then add three more opposite them to represent the negative axes. When you examine the Zome ball with these six positive and negative axes, they divide the ball into eight octants that, apart from sign and labeling of the axes, are equivalent. Thus we really need only examine the holes in the Zome ball that fall into one octant or boundary of that octant. A quick look will make it obvious that in each octant there is essentially only two kinds of holes into which a blue strut can be inserted (an axis and a non-axis blue hole), two into which a yellow can be inserted (in one of the planes determined by two axes and one making equal angles with all the axes, and only one for the red struts (again in a plane determined by two of the axes).

We need to show that if we assume that the Zome ball has coordinates $(0, 0, 0)$ the other end of each of those five types of struts will have coordinates satisfying Equation 2.

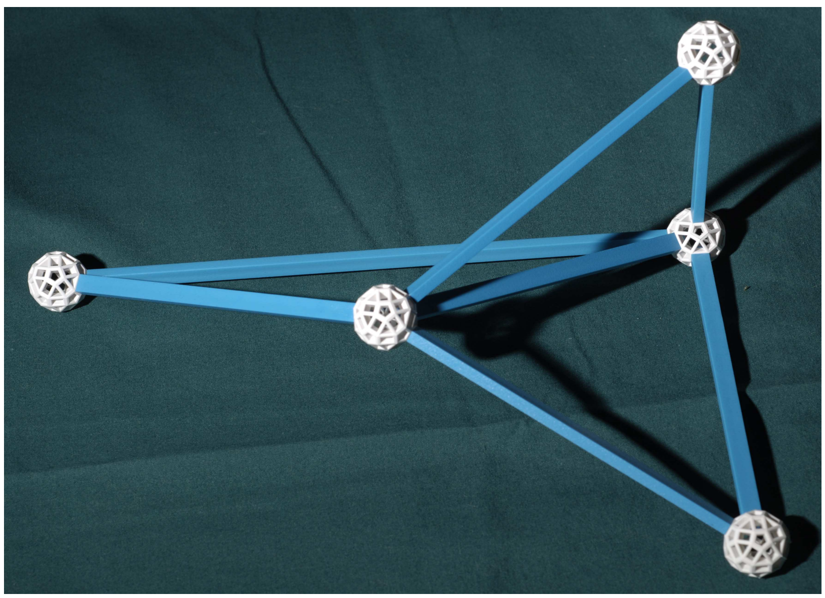

Figure 6: Blue Struts

First note that if any particular strut satisfies these conditions, then all struts of the same color inserted into the same hole will work, since every strut's length is a power of τ times another strut of the same color, and we know that multiplying coordinates by a power of τ can always be reduced to a value that is linear in 1 and τ as we showed in Section 4.

The easiest struts to consider are the blue axial struts. With a short blue, the coordinates will always look something like $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ or $(0, 0, \pm 1)$, and these all satisfy the conditions of equation 2.

The red and yellow struts that lie in a plane determined by a pair of axes are also quite easy. In fact, Figure 3 shows that a Y_2 strut from the origin has coordinates $\pm 1/2$, $\pm \tau^2/2 = (1+\tau)/2$, and 0 in some order, depending on the particular pair of axes that determine the plane

Figure 4 gives a similar argument that the R_1 struts have coordinates $\pm 1/2$, $\pm \tau$ and 0, in some order.

A yellow strut that makes equal angles with all three axes is shown in Figure 5. If you construct it with three B_1 axes and three Y_1 struts as shown, it is clear that the ball in the center of the yellow struts has coordinates $(\pm 1/2, \pm 1/2, \pm 1/2)$.

Finally, the trickiest standard strut turns out to be a blue strut in one of the off-axis holes.

In Figure 6, the coordinate axes are made of a B_1, a B_2 and a B_3. From the illustration, it is clear that the central ball, relative to the origin, has coordinates $\pm 1/2$, $\pm\tau/2$, and $\pm\tau^2/2 = \pm(1+\tau)/2$, in some order.

An interesting consequence of this is that if we had blue struts in every length, including $1/\tau$, $1/\tau^2$, $\ldots$, and if all we cared about were the locations of the Zome balls in the final structure, then there would be no need for the yellow, red or green balls since simple movements in the directions of the coordinate axes with these blue struts can get us to any ball location reachable using the red, yellow and green struts.

6 Appendix

Relative strut lengths:

In what follows, “HB” stands for “half blue”, “G” for “green”, “HG” for “half green” and “BG” for “blue-green”.

$\tau = \frac{1+\sqrt{5}}{2} = 1.6180339887498948482045868\,34$
$\sin 60^\circ = 0.8660254037844386467637231707$
$\sin 72^\circ = 0.9510565162951535721164393333$

$B_1 = 1$
$B_2 = \tau$
$B_3 = \tau^2$

$Y_1 = \frac{\sqrt{3}}{2} = \sin 60^\circ$
$Y_2 = \tau\frac{\sqrt{3}}{2} = \tau \sin 60^\circ$
$Y_3 = \tau^2\frac{\sqrt{3}}{2} = \tau^2 \sin 60^\circ$

$R_1 = \frac{\sqrt{2+\tau}}{2} = \sin 72^\circ$
$R_2 = \tau\frac{\sqrt{2+\tau}}{2} = \tau \sin 72^\circ$
$R_3 = \tau^2\frac{\sqrt{2+\tau}}{2} = \tau^2 \sin 72^\circ$
$HB_2 = \frac{\tau}{2}$
$HB_3 = \frac{\tau^2}{2}$
$HB_4 = \frac{\tau^3}{2}$

$G_0 = \frac{\sqrt{2}}{\tau}$
$G_1 = \sqrt{2}$
$G_2 = \tau\sqrt{2}$

$HG_1 = \frac{\sqrt{2}}{2}$
$HG_2 = \tau\frac{\sqrt{2}}{2}$
$HG_3 = \tau^2\frac{\sqrt{2}}{2}$

$BG_1 = 1$
$BG_2 = \tau$
$BG_3 = \tau^2$

Some of the information in this article is based on the web page called Analytic Zome:

`http://www.rawbw.com/ davidm/zome/`