Units

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1 Introduction

In this document we'll consider the many different aspects of units. Since the author is from the United States where the country is still afraid to go metric, many (most?) of the examples will be using the non-metric units.

We'll talk about conversion of units, definitions of units, dimensional analysis, and many other things.

2 What Are Units?

In arithmetic and algebra we're used to combining numbers in various ways, but as soon as we consider a real world problem, units come into play. As we all know, you can't add apples and oranges, and that's what units are about: to make sure that the mathematical operations make physical sense. The nice thing, though, is that you can use mathematical techniques to make sure the units make sense.

The great thing about units is that equations like the simple, "distance = rate times time":

$$d = rt$$

is true as long as the distance is some length, the rate is some velocity, and the time is some time. If the units of r are furlongs per fortnight, and the time is in milliseconds, the resulting distance makes sense, but has the somewhat unusual units "furlong-milliseconds per fortnight." Of course some work needs to be done so that these units make sense to us, but, hard as they are to understand, they are correct.

In this paper we'll chiefly look at units for physical measurements, but the same techniques work for any types of units: dollars, mouse clicks, candy bars, and even apples and oranges.

3 Unit Conversion

In this section we'll see how to deal with units like "furlong-milliseconds per fortnight," but we'll begin with simpler examples.

When two units basically measure the same thing, conversion is easy. For example, miles, feet, kilometers, and light years all measure the same thing; namely, length. If you know the conversion factor, you just need to multiply or divide by that to perform the conversion.

There is a totally naive approach. For example, if you know that 1 kilometer equals 0.621371 miles and you wonder long long a marathon (26.21875 miles) is in kilometers, you need to multiply or divide by 0.621371. Since you know that miles are longer than kilometers, there will be more kilometers than miles, so we need to divide 26.21875 by 0.621371 to make it bigger. (Multiplying by a number smaller than 1 would make it smaller. Thus a marathon's length is $26.21875/0.621371 = 42.194988$ kilometers.

This method works fine if you are quite familiar with the relative sizes of the two units in question, but if you're a little vague about the relative lengths of a mile and a kilometer, this can get you into trouble. Here is a completely mechanical way to do unit conversions with almost no chance of error. The beauty of this method is that it also works for more complex transformations, like miles/gallon to kilometers/liter.

It is based on the mathematical idea that you can multiply anything by 1 and that will leave it unchanged. Using the miles to kilometers problem, we can convert the equation:

$$1 \text{ kilometer} = 0.621371 \text{ mile}$$

to either of the following[1]:

$$\frac{1 \text{ kilometer}}{0.621371 \text{ mile}} = 1 = \frac{0.621371 \text{ mile}}{1 \text{ kilometer}},$$

giving us two different versions of the number 1. If we have a distance in miles, and multiply it by the version of 1 on the left, the "miles" in the numerator and denominator will cancel, leaving only kilometers. If we multiplied by the version of 1 on the right, it would still be technically correct, but the units would be:

$$\frac{\text{mile}^2}{\text{kilometer}},$$

which isn't particularly useful.

One nice result of this completely mechanical conversion is when it is harder to see whether the result will be larger or smaller in the new units. As an example, suppose gasoline costs 1.2 euros per liter and you'd like to see what that amounts to in dollars per gallon. Suppose the conversion factors are as follows[2]:

$$\begin{aligned} 1 \text{ dollar} &= 0.92 \text{ euro} \\ 1 \text{ gallon} &= 3.78541 \text{ liter}. \end{aligned}$$

When we have two equal quantities, we can write the 1 as a fraction with either of them in the numerator, so for conversion, we simply choose the form we need. In our example, we begin with the euro-per-liter value and convert it to its dollar-per-gallon form. Since we begin with "euro" in the numerator, we'll need to multiply the dollar-euro fraction with the "euro" in the denominator, and vice-versa for the gallon-liter fraction. The result we need is thus:

$$1.2 \,\frac{\text{euro}}{\text{liter}} \cdot \frac{1 \text{ dollar}}{0.92 \text{ euro}} \cdot \frac{3.78541 \text{ liter}}{1 \text{ gallon}} = 4.93749 \,\frac{\text{dollar}}{\text{gallon}}.$$

You can do a conversion gradually if you know a series of relationships to get you there. As a somewhat bizarre example, suppose you want to know how many centimeters (cm) there are in a mile (mi). You know there are 2.54 centimeters per inch (in), there are 12 inches in a foot (ft) and 5280 feet in a mile. Here's the conversion:

$$1 \text{ mi} \cdot \frac{5280 \text{ ft}}{\text{mi}} \cdot \frac{12 \text{ in}}{\text{ft}} \cdot \frac{2.54 \text{ cm}}{\text{in}} = 160934.4 \text{ cm}.$$

If you don't have access to the internet or other sources of information, you can memorize just a few conversion factors and using only those, be able to do a large number of conversions. The author knows the relations between inches, feet, yards, miles, and also between any of the metric lengh units. With one additional fact, that 2.54 cm is 1 inch, all possible conversions are possible, with the correct string of calculations. If you know that 1 liter is about 1.057 quarts, you can convert most volumes, et cetera. Similarly, 1 kilogram equals about 2.2 pounds makes most masses[3] interconvertible.

[1]Note that when we use English to speak about these relationships sometimes we use the singluar and sometimes plural ("one mile per gallon," versus "two miles per gallon"). The singular/plural difference is due to English, not the units, so we will always write units in a singluar form here.

[2]Gallons and liters are at a fixed relationship, but the dollar-to-euro value varies.

[3]Well, thinking of a pound as a mass, at least.

4 Unit Sizes and Origins

We humans aren't great at understanding very large or very tiny numbers. That's why it's easiest to understand measurements when the units are in not too big a range around one. In fact, it's probably better to have measurements that are larger than one; it's not too hard to think about 150 feet, and much more difficult to think about what 0.0066666 feet means. (The 0.0066666 is about $1/150$ which is about as far below 1 as 150 is above.)

Thus we talk about road trips in miles, heights of people in inches, distances between stars in light-years or parsecs, astronomical unit, and distances within an atom in angstroms (Å).

Long ago when it was not critical to obtain exact results, most of the units were based on practical and somewhat arbitrary considerations. A "foot" might be the length of the current king's foot. An acre was the amount of land that could be plowed in one day by a yoke of oxen, a mile was a thousand steps, and so on.

5 Physical Measurements

It is somewhat amazing how many of the things measured by physicists and engineers have dimensions that are obtained from some basic types of unit. We'll consider three of them: length, mass, and time[4]. All length units are "the same" in the sense that they measure the same sort of thing and they're easily converted to other units via a multiplication by a constant that measures the ratios of the sizes of the units. To convert from miles to yards, multiply by 1760 and so on. The same is true for the other units as well.

5.1 Unit Types and Derived Units

There is, however, something very different about the unit types. It makes no sense to convert feet to seconds or kilograms to miles. But there is a tremendous value in being able to produce compound units. For now, let's not worry about the particular unit of length, mass, or time we are using, and we'll just use L, M, and T, respectively, to stand for that sort of unit. These fundamental units are called "base units."

If we wish to talk about area, for example, the unit type will be L^2: a length times a length. Every area will have that sort of unit. Almost always we square the same sort of unit, as in "square inches," "square miles," et cetera. An area unit like "acre" has L^2-type units although the name buries that fact. One of the original official descriptions of an acre (after the oxen-work-based definition) was an area on land that was one chain (22 yards) long and one furlong (220 yards) wide, so it is of type L^2 although the two different lengths in the definition have different units.

Unit types that involve combinations of the base units are called "derived units."

Similarly, volume has units of type L^3, usually cubic feet or cubic inches or similar, but water is often measured in a more convenient form as gallons or acre-feet. Acre-feet is convenient since if your 3-acre lake gains one foot in depth, you've added 3 acre-feet of water to it; it's a block of water 1 chain by 1 furlong by 1 foot, which gives the three L's in the L^3 form.

Units like gallons or barrels are really volumes and can be converted to cubic inches or whatever. They intrinsically have units of type L^3.

Here are some compound unit types constructed from the primitive unit types:

- L^2: Area
- L^3: Volume
- LT^{-1}: Velocity—change of distance per unit of time.
- LT^{-2}: Acceleration—change of velocity per unit of time.

[4]There are others, like charge, but we'll just use these three for our examples.

- LT^{-3}: Jerk—change of acceleration per unit of time.
- MLT^{-1}: Momentum—mass times velocity.
- MLT^{-2}: Force—mass times acceleration.
- $ML^{-1}T^{-2}$: Pressure—force per unit area.
- ML^{-3}: Density—mass per unit volume.
- ML^2T^{-2}: Work (or Energy)—force applied over a distance.
- ML^2T^{-3}: Power—energy used per unit of time.
- T^{-1}: Frequency—number of occurrences per unit of time.

As an example, let's check that energy always has units of the form ML^2/T^2. Here are a few formulas used to compute energy:

- The potential energy of an object is given by mgh, where m is the mass, g is (the acceleration of) gravity, and h is the height above the ground. A mass, an acceleration and a length multiply to give the correct units.
- You pay your electric (energy) bill in (kilo)watt-hours. A watt is a measure of power and an hour, of time. Again, the product of those unit types yields an energy.
- Einstein said that $E = mc^2$. m is mass and c is a velocity (the velocity of light). The product works out to be energy units.

The nice thing about thinking about units in this way is that the formulas of physics work no matter what the units are. To take an example from elementary school mathematics, consider the formula:

$$d = v \cdot t,$$

distance equals velocity multiplied by time. v could be measured in miles/hour and time in years and the formula is still true. A brute-force multiplication of those values would give you a distance in terms of mile-(year/hour). The unit (year/hour) has no unit type, but if we know the number of hours in a year[5]:

$$1 \text{ year} = 8760 \text{ hour},$$

We can multiply our answer by:

$$1 = \frac{8760 \text{ hour}}{\text{year}}$$

and obtain the result in miles.

One final thing: in what follows we will be a little sloppy with the English "pound" weight. A pound is a weight, which is a force, not a mass, so to be precise, there is no way to convert between kilograms and pounds, although you see the conversion:

$$2.2 \text{ pounds} = 1 \text{ kilogram}$$

in many places.

A mass will weigh different amounts on different planets, but will always have the same mass. When we are being sloppy, we may consider a pound to be that mass which has a weight of one pound on Earth. See Section 5.3 for the technically correct way to handle mass in the English system.

[5]This is a 365-day year.

5.2 SI Units: The Metric System

The "SI" stands for "Système International (d'unités)." It uses a particular specific unit for each of the unit types. There are seven base units (in the previous section we considered only three) which are: length, mass, time, electric current, thermodynamic temperature, amount of substance (just a number), and luminous intensity. The units used for those unit types are, respectively, the meter, kilogram, second, ampere, degree Kelvin, mole, and candela. It's an extension of the MKSA system (meter, kilogram, second, ampere)[6].

For now we'll ignore all but the first three types, but the system gives names for most of the derived units. Here are some examples, where we will use the forms "kg" for kilogram, "m" for meter, and "s" for second:

- $1\ \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$: Newton; unit of force.
- $1\ \mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}$: Pascal; unit of pressure.
- $1\ \mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{s}^{-2}$: Joule; unit of energy (or work).
- $1\ \mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{s}^{-3}$: Watt; unit of power.

There are lots of others.

The system also defines the various multiples of 10 used to modify the units. For example, we call 1000 watts a "kilowatt," and a "kilo" prefix always means "multiply by 1000," "micro" means "divide by 1000000", et cetera. Here's a complete list of the prefixes for SI units:

Multiplier	Prefix	Multiplier	Prefix
10^1	deca	10^{-1}	deci
10^2	hecto	10^{-2}	centi
10^3	kilo	10^{-3}	milli
10^6	mega	10^{-6}	micro
10^9	giga	10^{-9}	nano
10^{12}	tera	10^{-12}	pico
10^{15}	peta	10^{-15}	femto
10^{18}	exa	10^{-18}	atto
10^{21}	zetta	10^{-21}	zepto
10^{24}	yotta	10^{-24}	yocto

So in 20 grams of hydrogen gas, there are about 10 moles of H_2 molecules, or $10 \cdot 6.023 \times 10^{23} = 6.023 \times 10^{24} = 6.023$ yotta molecules.

The metric system seems way more logical than the English system in that the basic units differ in size by factors of 10 in every case. There is one example from the English system that is much older (13^{th} century) that's equally good. It is the system for liquid measure:

2 gills = 1 chopin	2 demibushels = 1 bushel or 1 firkin
2 chopins = 1 pint	2 firkins = 1 kilderkin
2 pints = 1 quart	2 kilderkins = 1 barrel
2 quarts = 1 pottle	2 barrels = 1 hogshead
1 pottles = 1 gallon	2 hogsheads = 1 pipe
2 gallons = 1 peck	2 pipes = 1 tun
2 pecks = 1 demibushel	

[6]Another system of units is also popular if you're working with smaller measurements: CGS, standing for "centimeter, gram, second."

5.3 The Slug

In English units, we measure weight in pounds, but a pound is a force, not a mass. So questions like, ”What would a man who weighs 150 pounds on Earth weigh on Jupiter?” make sense. A force is a mass times an acceleration (so in the example above, the difference in weights is due to the difference in the gravitational accelerations between Earth and Jupiter).

The English unit of mass is called a “slug” and in terms of weight (in pounds, on Earth) the slug’s units are:

$$\frac{\mathrm{lb\ s}^2}{\mathrm{ft}}.$$

Since the gravitational acceleration on Earth is about 32.17 that makes a slug equal to 14.59 kg.

Thus we can easily, if we wish, convert all our energy values into slug-furlongs-squared per fortnight-squared.

5.4 Temperature

An early measurement of temperature was proposed by Daniel Fahrenheit in 1724. It defined 100 degrees as roughly human body temperature and zero degrees as the coldest temperature he could achieve in a lab (a freezing brine of saturated ammonium chloride). Body temperature isn’t exactly constant and much lower temperatures a achievable so the definition eventually became defined by $32°$ and $212°$ are the temperatures at which water freezes and boils at standard pressure at sea level.

Celsius temperatures used the same two points, but defined them as $0°$ and $100°$. A linear conversion is a little messier than with most units, since the zero points don’t match. (This isn’t a problem with most measurements since zero feet is the same as zero meters is the same as zero light-years.) In any case, here’s the conversion:

$$C = \frac{5}{9}(F - 32),$$

where C and F are the temperatures in Celsius and Fahrenheit, respectively.

A more modern and easier to measure zero-point for temperatures Celsius are based not on the freezing point at standard pressure at sea level, but on the triple point of pure water: the temperature/pressure at which ice, liquid water, and water vapor co-exist.

Some physical experimentation indicates that the ideal gas law relating the temperature T (Celsius), the pressure P, the volume V and the number of moles of the gas n are related as follows with an appropriately chosen constant R:

$$pV = nR(T + 273.15).$$

This sort of indicates that the zero point of temperature was chosen wrong, and in fact, we define a temperature of absolute zero as $-273.15°$ Celsius.

If we measure temperature from absolute zero and use steps equal in size to the steps in the Celsius scale we call this temperature the Kelvin scale at which absolute zero has temperature zero and water freezes at $273.15°$. If the temperature is measured in degrees Kelvin, the ideal gas law looks more reasonable:

$$pV = nRT.$$

If you prefer Fahrenheit-sized degrees, there’s a corresponding Rankine temperature scale where zero degrees is again absolute zero and water freezes at $459.67°$. The ideal gas law is the same, but with a different constant R to account for the differences in degree size.

5.5 Angle

Most measures of angle simply describe the angle in terms of a fraction of a complete turn. The common 360° measure for a complete turn allows us to break it into 360 equal parts. There's nothing magical about 360; it's just that this number is divisible by a lot of useful numbers: $1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, \ldots$, so we can talk about a half, a third, a fourth, …, of a circle as whole numbers of degrees. Also, there's a possibility that the 360 is related to the number of days in a year (which wasn't measured all that accurately thousands of years ago).

The division of degrees into minutes and seconds (60^{ths} and 3600^{ths} of a degree) is also ancient, and 60 has many of the nice divisibility properties of 360. In fact, sometimes the seconds are divided into 60^{ths} which are called "tertias."

Of course many methods are used for geographic coordinates (latitude and longitude) sometimes it's just degrees and decimals of a degree, like 122.71231°, sometimes it's degrees, minutes, and seconds, like $122^\circ\ 42'\ 31''$ and sometimes (as for example what's used in the Geocaching game) a mixture: $122^\circ\ 42.1234'$.

To interconvert decimal versions of angles and the degrees-minutes-seconds versions is not too hard. Let's do it both ways. If we need to convert 47.1234°, we know the answer will still have 47° so we just need to work on the .1234 part:

$$.1234 = m/60,$$

or $m = 60 \cdot .1234 = 7.404$, so we have 7 minutes with a fraction of .404 minutes. Just do the same thing to the fraction of minutes to obtain seconds:

$$.404 = s/60,$$

so $s = 24.24$. The full conversion, therefore, is: $47.1234^\circ = 47^\circ\ 7'\ 24.24''$.

Let's do the conversion in the opposite direction and we'll begin with our previous answer so we'll have a check of our results. Convert $47.1234^\circ = 47^\circ\ 7'\ 24.24''$ to degrees with a decimal part.

The (fractional) part of the minutes is obtained by dividing the number of seconds by 60, giving $24.24/60 = .404$ minutes. Add this to the 7 minutes we have to obtain 7.404 minutes. That's $7.404/60 = .1234$ of a degree, so the final result is 47.1234°.

When the metric system was introduced there was a movement to change right angles to have 100 units instead of 90, and the new unit is called a "gradian" or "gon".

For mathematical uses, the radian is a more useful measure of angle. The idea is that with a circle of radius 1 any angle will subtend a certain length of arc, and that length is the (radian) measure. If the circle were double the size, the length of the arc would double, so the radian measure is basically a ratio of lengths: the length along the circle divided by the length of the radius. This perhaps provides an intuitive way to see that angle measure is basically unitless: it's a length divided by a length.

So angles are sort of unitless and sort of not: for the purposes of exponentials and trigonometric functions, they behave as unitless, but you sure have to know the type of angle before you calculate its cosine.

If you've already memorized a few digits of π you don't have to memorize the conversion factor to get from degrees to radians and vice-versa:

$$\pi \text{ radians} = 180^\circ,$$

and you can work out the conversion factor from that.

For some purposes it's good to allow angles restricted to be between 0° and 360° (or 0 and 2π radians), and for other uses, large positive and negative values make sense. For direction, the restricted values are fine, but although an ice skater who has just spun for 3600° will probably be a lot dizzier than one who didn't turn at all, even though they both wound up facing the same direction.

The grade (or slope, incline, pitch) is the tangent of the angle the road (or whatever) makes with the horizontal. This is the usual "rise/run" used in mathematics and is often measured in percent, so a road that climbs at a 10% grade gains one foot in altitude for every 10 feet of horizonal movement. Grades vary from zero (flat) to infinity (a sheer cliff).

6 Keeping Track of Units

When the author took a physics class in college, he learned about one of the huge advantages of algebra over arithmetic. To illustrate, we'll use a very simple example from high-school physics to illustrate the method. Suppose that the class has just covered the formula for the height of an object above the ground if it started at altitude d, had an initial velocity toward the ground of v (v would be negative if the initial velocity were up) and the acceleration of gravity is g. Ignoring air resistance, et cetera, here's the formula for the height $h(t)$ of the object above the ground at time t:

$$h(t) = -gt^2/2 + vt + d.$$

Suppose the problem is this: an object is initially thrown upward at 17 meters per second from a point 53 meters above the ground. How many seconds later will the object hit the ground?

In high school, I did this, since I knew that the acceleration of gravity on Earth is 9.8 meters per second per second:

$$h(t) = -4.9t^2 - 17t + 53 = 0,$$

and then I solved for t using the quadratic formula. This wasn't particularly dangerous since there was only one step (using the quadratic formula) to obtain the answer from the given data. (Of course I needed to check that all the units were in the meter-kilogram-second units, but once they were, I knew the value of t would be in seconds.

In college, however, the professor would have worked it like this: She'd say, "Ok, let $v = -17$ and $d = 53$. we have:

$$h(t) = -gt^2/2 + vt + d = 0.$$

Use the quadratic equation to solve for t and obtain:

$$t = \frac{-v \pm \sqrt{v^2 + 2gd}}{g}.$$

(Again, since it's a high-school type problem the answer took just one step, but imagine that we had to do a full page of calculation to obtain t.)

We can now just plug in the numbers for v, g and d to obtain the result, but this nice thing here is that before doing so, we can sanity-check our calculations by making sure the the units of the answer come out as time. The numerator combines a velocity with the square root of a velocity-squared and an acceleration times a lengh (which is also a velocity squared), and the square root of a velocity squared is also a velocity. Thus all the terms in the numerator are velocities (type L/T). The denominator is an acceleration with type L/T^2, so the quotient has type T, as desired.

The thing that surprised me at first was the idea of converting known numbers to letters which seemed counter-intuitive, but look at the advantage: the letters drag the units along with them to provide a check on the rest of your calculations.

7 Logarithms and Exponentials

The equations describing some physical processes involve things like trigonometric functions, exponentials, logarithms, et cetera. But what does it mean, say, to take the logarithm of a mass or length? It makes no sense, so that means that whenever one of these functions appears in a physical formula, the value passed to the function must be dimensionless.

That means that whatever combination of units appears in such a situation includes either a physical constant or some constant of the situation being described that causes the units to cancel to be something dimensionless.

7.1 Radioactive Half Life

If half of a substance decays in a certain amount of time, say $t_{1/2}$ then the amount left after time t is:

$$A(t) = A_0 2{-}t/t_{1/2},$$

where A_0 is the original amount. Note that the exponent of 2 has dimensions time divided by time, or, in other words, is unitless.

Notice that before performing the exponentiation, you need to get rid of the units entirely. For example, if the half life is one minute and you want to see how much is left after 30 seconds, the exponent is 30 seconds/minute. Since seconds/minute isn't dimensionless, you need to multiply by 1, which is $1/60$ minute/second, yielding an exponent of $1/2$.

The same idea applies to exponential growth (of a population, perhaps). If we start with a population P_0 that doubles every t_2 years then the population $P(t)$ at time t is given by:

$$P(t) = P_0 2^{t/t_2}.$$

The exponent is positive because it's growing, but the t/t_2 in the exponent is unitless.

7.2 Harmonic Oscillator

If a voltage or a force has a circular component, the equation will consist of a sine or cosine or exponential, sort of like this:

$$\begin{aligned} F &= F_0 \cos\, \omega t \\ V &= V_0\, e^{i\omega t} \end{aligned}$$

then t will be a time, and ω must have unit type $1/T$, and in fact, ω is just a frequency (like cycles/second).

Note that exponential formulas can be converted to logarithms or inverse trigonometric functions. The equations above can be converted to:

$$\begin{aligned} \arccos(F/F_0) &= \omega t \\ \log(V/V_0) &= i\omega t. \end{aligned}$$

It's clear that a force divided by a force or a voltage divided by a voltage are unitless measurements.

7.3 The Boltzmann Distribution

In a system where the particles can have various energy states, then as the temperature varies, the distribution of the particles' energies $F(E)$, can be expressed as:

$$F(E) \propto e^{-E/kT},$$

so the ratio of the distributions for two different energies E_1 and E_2 is given by:

$$\frac{F(E_1)}{F(E_2)} = e^{(E_2 - E_1)/kT}.$$

T is temperature (Kelvin) and so since the numerator of the exponent is an energy the denominator must be also to eliminate the dimensions, so the Boltzmann constant k must have units energy divided by temperature.

7.4 Entropy

If a system has n possible states for its particles, and the probability of being in state i is p_i, then the entropy S of the system is proportional to:

$$S \propto -\sum_{1}^{n} p_i \log p_i.$$

Probabilities are dimensionless, so we're feeding dimensionless measurements to the logarithm function.

8 Dimensional Analysis

Sometimes you can figure out approximately what a formula should look like, based only on the units. As an example, suppose that you would like to know what the period is of a simple pendulum. We make the usual assumptions: all the mass is in the bob and the string is massless. It seems clear that the things that might affect the period (the time for it to swing back and forth) are the length of the string, the mass of the bob, and the strength of gravity.

Gravity, g, is an acceleration (L/T^2), mass, m, is a mass (M), and the length of the string, l, is a length (L). We need to combine these three things into a formula that yields a time (T). At first it seems weird, but we can just combine the acceleration and length as l/g and obtain units of T^2; the mass doesn't seem to be required at all. Thus a possible form for the equation for the period t might be something like:

$$t \propto \sqrt{\frac{l}{g}}.$$

We use the symbol "$\propto$" (proportional to) since there can be a dimensionless constant k involved, but a good guess for the correct formula is this:

$$t \approx k\sqrt{\frac{l}{g}},$$

for some constant k.

In fact the formula turns out to be this:

$$t \approx 2\pi\sqrt{\frac{l}{g}},$$

and in fact, the period doesn't depend on the mass at all!

9 Special Relativity

Without going into details, most people know a little bit about Einstein's special relativity: that in a moving frame of reference, lengths change and time appears to run at a different speed than they do in a fixed frame of reference.

For example, if you are on the earth and measure the distance between a pair of objects and find that it is one meter, a person passing by in a spaceship at half the speed of light will see those objects closer together than one meter.

In three-dimensional Euclidean space, if we have assigned x, y, and z coordinates and we have two objects at (x_1, y_1, z_1) and at (x_2, y_2, z_2) then we can calculate the distance D between them as follows:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

which is basically the Pythagorean theorem in three dimensions.

The nice thing about the formula above is that the value of D will be exactly the same, even if you use a completely different Cartesian coordinate system. The distance between two objects is independent of the coordinate system they're described in.

9.1 The Invariance of the Interval

Of course since our friend in the rapidly-moving spaceship sees a different D that means that her coordinate system must look different. So in relativity theory, there is no such thing as an absolute distance. However, there *is* something that is absolute in all frames of reference, including those at rest, or those for observers traveling at any speed and in any direction.

Consider two events: things that occur at a specific point and at a specific time. There is a mathematical measurement of a quantity called the "interval" between the two events and it combines both the physical distance between the events and the time between the events. The coordinates for an event include not only a place, but a time, so four-dimensional coordinates are required to identify both the time and place of any event. Let's say that the events occur at (x_1, y_1, z_1, t_1) and at (x_2, y_2, z_2, t_2), where t_1 and t_2 are the times the events occur in a particular frame of reference. Then the following is a formula for the interval, I between the events:

$$I = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - k^2(t_2 - t_1)^2},$$

where k is a constant.

Without the k the dimensions are messed up. We'd be combining a measurement with an L^2 unit type with a measurement having a T^2 unit type and that doesn't make sense. Thus k must be a constant that converts a T to a L, and that must be L/T, so k is basically a velocity. In fact, $k = c$, where c is the speed of light, so the true formula for the interval between two events is:

$$I = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2}.$$

Nice, huh?

9.2 Adding Velocities

Another example from special relativity is the fact that velocities do not add. For low speeds they *almost* do, but at high speeds, the error you obtain by just adding them gets larger.

Imagine the following situation: You are on the ground, fixed, and a spaceship goes by at $3/4$ of the speed of light: $\frac{3}{4}c$. A guy in the spaceship at the same time sees another spaceship travelling in the same direction, and it appears to him that this second spaceship is moving (relative to him) at the same $\frac{3}{4}c$. How fast does the second spaceship appear to be moving to you, on the ground?

If you just add the two velocities, you obtain $1.5c$: one and one-half times the speed of light, but it's well-known that objects cannot travel faster than c relative to each other, no matter what the frame of reference.

Here is the correct formula, according to special relativity for adding two velocities v_1 and v_2 in the same direction, as in the example above:

$$v_{1+2} = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}.$$

(Notice that the denominator is unitless, as required, since the numerator is already a velocity.)

In the example above, $v_1 = v_2 = 3c/4$, so the apparent velocity of the faster ship from the point of view of the guy on the ground is:

$$\frac{1.5v}{1 + 9/16} = \frac{24}{25}c,$$

so it's still not faster than light.

To obtain an estimate of what the error amounts to for speeds we're used to, imaging two cars: the guy on the ground first sees the first going 100 km/hr and the guy in that car sees the second car moving at the same 100 km/hr relative to him. How fast does the guy on the ground see the second car moving? It's not quite 200 km/hr, but how much less, actually?

The speed of light is approximately $c \approx 300,000$ km/second, so we'll use that approximation.

We need to convert c into km/hour, and that gives $c = 1,080,000,000$ km/hour. The denominator of the formula will be $1 + \frac{100 \cdot 100}{1080000000^2}$. A little arithmetic tells us that the correct apparent speed is:

$$199.99999999998285 \frac{\mathrm{km}}{\mathrm{hr}}.$$

This is why relativity isn't particularly noticeable in our everyday lives.

10 Definitions of Basic Units

We'll mostly consider just the base units in the metric system here.

10.1 Length

One proposed way to obtain a unit of length was as the length of a pendulum that would go through a complete cycle in one second. Gravity varies over the surface of the earth (slightly), and in any case the accuracy of a meter can only be determined with the same accuracy as a second can be measured.

Originally, a meter was defined to be $1/10,000,000$ of the distance from the north pole to the equator.

Since this is hard to measure, and depends on lots of things, the definition was changed in about 1870 to be the distance between two lines on a standard bar composed of an alloy of 90% platinum and 10% iridium, measured at the melting point of ice.

In 1960 it was changed to be $1,650,763.73$ wavelengths of the orange-red emission line in the electromagnetic spectrum of the krypton-86 atom in a vacuum.

In 1983 it was changed to be the length of the path travelled by light in vacuum during a time interval of $1/299,792,458$ of a second.

We are warned, however, to use this definition only when the distances measured are small compared to the curvature of space described by general relativity.

This is unrelated to the meter, but here's an interesting fact about the inch. From one of the original definitions as the length of three barleycorns, over time better and better definitions of the inch were used, and a very accurate definition that depended on the accuracy of the meter was that 1 inch is $1/39.37$ meters. This made the inch have a length of 2.54000508 centimeters. Since this is so close to 2.54, in 1959 the international inch was changed to be exactly 2.54 centimeters and the United States joined in that treaty, but we use the old definition for surveying work, where the difference amounts to about $1/8$ inch per mile, or about 10 meters across the width of the United States.

For much larger and smaller distances, it's useful to have much larger or smaller units to measure them. Light year = $yr \cdot c$, parsec, angstrom = $10^{-10}m$.

10.2 Time

A second was originally defined as the appropriate fraction of a mean solar day with 24 hours per day, 60 minutes per hour and 60 seconds per minute. This isn't bad, but with the earth gradually slowing down, it's not accurate enough for modern purposes.

To take into account the slowing of the earth, a better definition was this: ”the fraction 1/31,556,925.9747 of the tropical year for 1900 January 0 at 12 hours ephemeris time.” This definition was used up until 1960.

In 1960, the definition of a second was changed to be ”the duration of 9192631770 cycles of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom,” and in 1997, this definition was slightly changed to ”refer to a cesium atom at rest at a temperature of 0 K.”

10.3 Mass

The kilogram was originally defined to be the mass of one liter of water at the melting point of water. In 1799 a prototype was made and later found to be slightly off, so the official definition became the mass of 1.000028 liter of water at its greatest density.

There are copies of the prototype kilogram, and over time, their relative weights vary, so it has been proposed (but as of 2014 not acted upon) that the kilogram be based on a physical constant of nature (Planck's constant).

There are other proposals, including one that is based on a certain count of a particular kind of atom: ”$83\ 1/3$ moles of Carbon 12.” A mole is defined to be about $6.02214162 \times 10^{23}$ atoms.

There is also the “Avogadro Project” that proposes to use ultra-pure silicon for the definition of a kilogram. A sphere with an accuracy of 0.3 nanometers can be formed from a perfect crystal of silicon. That amounts to an error of about one atom-width.

Here's another proposal: ”a kilogram is the mass which would be accelerated at precisely $2\times10^{-7}\mathrm{m/s}^2$ when subjected to the per-meter force between two straight parallel conductors of infinite length, of negligible circular cross section, placed one meter apart in vacuum, through which flow a constant current of $1/(1.60217 \times 10^{-19})$ elementary charges per second”.

11 Energy, Work, Heat, and Power

Energy will always have unit type ML^2T^{-2}, and sometimes it is called “energy,” sometimes “work,” and sometimes “heat.” They are all the same (but the terms are used in different situations.

In the pure MKSA system, a joule of energy is the amount of energy needed to push against one Newton of force for one meter. A watt is one joule per second. If you multiply watts by time, you get back to energy, so perhaps the best known units of energy by “the man in the street” is kilowatt-hours, since that's what you pay for with your electric bill. You're paying directly for your energy usage, even though the energy you got as a combination of heat and work and light was not measured by you as a force through a distance, but as a voltage pushing through a resistance. (The electrical energy may have been generated by a force through a distance (hydroelectric energy from water's weight pushing a generator's turbine).

Horsepower is a commonly-used unit of power; namely (supposedly), the power of a single horse. It is defined to be 550 foot-pounds per second, or 745.7 watts. The amount of work such a theoretical horse does would be measured in horsepower-hours or similar.

When we measure heat, we often use calories or Calories. A “calorie” is the amount of heat required to raise the temperature of one milliliter of water by one degree Celsius. Of course the precise definiton includes exact specifications for the temperature at which it's measured. When you eat an ice-cream bar, you eat, say, 300 Calories (note the upper-case “C”). A Calorie is equal to a kilocalorie, or 1000 calories. One calorie wouldn't keep a bug alive for very long.

One kilocalorie is equal to 4184 joules. If you're interested in losing weight through exercise and you wonder how much work you'll need to do on an exercise bike with a power meter to undo the damage of one Dove Bar (about 260 calories), the rough calculation is pretty easy. We humans are close to 25% efficient, meaning that to do one calorie of work as muscle output we need to burn about 4 calories of energy (from fat or elsewhere). Thus to produce 4184 joules, we need to burn about 4000 calories of fat, and since 4184 is almost the same as 4000 for our rough calculation, it comes out to be about one Calorie (kilocalorie) for each 1000 joules.

So if you pedal away at 200 watts, every 5 seconds you burn 1 Calorie. So to burn the Dove Bar, you need about $5 \times 260 = 1300$ seconds, or about 22 minutes!

Then there's the BTU (British Thermal Unit) that's used to measure furnaces sometimes. It's based on the same idea as the calorie, except that it's the amount of heat needed to raise the temperature of one pound of water by one degree Fahrenheit. It's about 1055 joules.

12 Physical Analogs

Let's consider two different physical systems: one that looks at the motion of an object in a straight line, with forces, masses, et cetera, and another that looks at rotations of a body about an axis in the plane. To make an object move in a straight line, you apply a force to it in the direction of the line. To make an object spin about an axis, you apply a force to some point on the object perpendicular to the line from the axis to that point.

When motion occurs along a line, the position (length) changes. When rotation occurs, the angle changes. We can measure the rate the position changes (the speed, or velocity) and we can measure how fast the rotating object turns (how fast the angle changes). In fact, for most of the physical measurements of the object moving in a line there is a corresponding measurement of the turning object and not only that, but the equations that describe the relationships are the same. So if you're an expert at solving equations of motion along a line, you are automatically an expert at solving the equations of rotation.

Here's a chart showing the correspondence:

Displacement	x	L	Angular Displacement	θ	1
Veocity	$v = dx/dt$	LT^{-1}	Angular Velocity	$\omega = d\theta/dt$	T^{-1}
Acceleration	$a = d^2x/dt^2$	LT^{-2}	Angular Acceleration	$\alpha = d^2\theta/dt^2$	T^{-2}
Mass	m	M	Moment of Inertia	I	ML^2
Force	ma	MLT^{-2}	Torque	$\tau = I\alpha$	ML^2T^{-2}
Work	$\int F\,dx$	ML^2T^{-2}	Work	$\int \tau\,d\theta$	ML^2T^{-2}
Kinetic Energy	$mv^2/2$	ML^2T^{-2}	Kinetic Energy	$I\omega^2/2$	ML^2T^{-2}
Power	Fv	ML^2T^{-3}	Power	$P = \tau\omega$	ML^2T^{-3}
Momentum	$p = mv$	MLT^{-1}	Angular Momentum	$L = I\omega$	ML^2T^{-1}

Here is a similar chart comparing mechanical and electrical properties:

Independent variable	time (t)	time (t)
Dependent variable	position (x)	charge (q)
Inertia	mass (m)	inductance (L)
Resistance	drag coefficient (γm)	resistance ($R = \gamma L$)
Stiffness	stiffness (k)	capacitance^{-1} ($1/C$)
Resonant frequency	$\omega_0^2 = k/m$	$\omega_0^2 = 1/LC$
Period	$t_o = 2\pi\sqrt{m/k}$	$t_0 = 2\pi\sqrt{LC}$
Figure of merit	$Q = \omega_0/\gamma$	$Q = \omega_0 L/R$

Here's a similar correspondence between electrical current and linear motion. Suppose you have a mass m connected to a spring with spring constant k and a resistive trem of γm acted on by a force $F(t)$. Then the position of the object $x(t)$ is given by the solution to the following differential equation:

$$F(t) = m\Big(\frac{d^2x(t)}{dt^2}\Big) + \gamma m\Big(\frac{dx(t)}{dt}\Big) + kx(t).$$

Now consider the charge $q(t)$ on a capacitor of capacitance C in a circuit with inductance L and resistance R where the voltage $V(t)$ is applied the amount of charge $q(t)$ at any time is given by the solution to the following differential equation:

$$V(t) = L\Big(\frac{d^2q(t)}{dt^2}\Big) + R\Big(\frac{dq(t)}{dt}\Big) + \frac{1}{C}q(t).$$

Notice that these are exactly the same differential equations, and you can check that the units in each term are the units of force (in the first one) and units of voltage (in the second).

The mechanical system is quite simplified, but one can write down similar differential equations for much more complicated linear mechanical systems and for every one, a corresponding electrical system can be built. The differential equations may be very hard to solve, so designing a mechanical system with desired properties would be hard to do and doing it by trial and error would be very expensive since changing masses, springs, et cetera, is difficult.

On the other hand, it's easy to make an electrical circuit that includes variable inductors, resistors and capacitances, so by just turning knobs, you can simulate continuously varying masses, spring constants, et cetera, to obtain the desired behavior of the amount of charge, $q(t)$. Such electrical systems that are built to simulate mechanical ones are known as analog computers.

13 Physical Constants

As far as we know, the following values are physical constants and do not vary in time or relative to location in the universe.

- c, the speed of light in a vacuum. It's approximately equal to $299,792,458$ m/s.
- h, Planck constant: Approximately $6.626070040 \times 10^{-34}$ joule seconds. Often it is useful to divide this by 2π to form another constant, $\hbar = h/2\pi$.
- e, the charge of an electron: Approximately $1.6021766208 \times 10^{-19}$ coulombs.
- ϵ_0, the electric constant or permittivity of free space: Its value is approximately $8.854187817 \times 10^{-12}$ farads/m.
- G, the gravitational constant. The gravitational force between two spherical objects having masses m_1 and m_2 with r the distance between their centers is given by $F = Gm_1m_2/r^2$. The value of G is about $6.674 \times 10^{11} Nm^2/kg^2$.

13.1 The Planck Length

If we combine The Planck constant, the gravitational constant and the speed of light properly we can obtain a pure length, called the "Planck length":

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616119 \times 10^{-35} m.$$

It's not clear what physical significance this has, but it is a fundamental length, defined solely on other fixed physical constants.

13.2 The Fine Structure Constant

If we write down the unit types of all the constants above, we obtain (including A for current in amperes):

- c: LT^{-1}
- h: ML^2T^{-1}
- e: AT
- ϵ_0: $T^4A^2L^{-3}M$

More useful in most physics applications is $\hbar = h/(2\pi)$, called the “reduced planck constant), but this has the same unit type as h.

The fine structure constant α is this:

$$\alpha = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c}.$$

It is constructed as a product of values that seem to be constant in the universe, so this value will also be constant everywhere in the universe. What is its unit type?

Here it is:

$$\frac{A^2T^2}{(T^4A^2L^{-3}M)(ML^2T^{-1})(LT^{-1})}.$$

If you work it out, everything cancels, so the fine structure constant is unitless! The value is about $7.297352566 \times 10^{-3}$ and the inverse is about 137.035999139. Nobody knows why this number should pop up, but it does, in many, many physics relationships.

14 Practice Problems

Do not do any arithmetic, except on the final problem. Just leave your answer in the form of a series of multiplications and divisions.

Using *only* the following conversion values, write down expressions which, when the arithmetic is performed, would yield the correct results. As an example, if a problem asks to find out how many inches there are in 3.7 feet, since we know 12 inches equals 1 foot, the solution would be "3.7×12 inches".

14.1 Conversion Values

Length	
12 inches	1 foot
1 yard	3 feet
5280 feet	1 mile
1 chain	22 yards
1 furlong	220 yards
39.37 inches	1 meter
Area	
1 acre	1 chain-furlong
Angle	
1 degree	60 minutes
1 minute	60 seconds

Volume	
7.48 gallon	1 foot^3
1 meter^3	1000 liter
1 board-foot	1 foot^2-inch
Time	
1 year	365 days
1 day	24 hours
1 hour	60 minutes
1 minute	60 seconds
Currency	
.92 euro	1 dollar
Mass	
1 kilogram	2.2 pounds

14.2 Exercises

1. Convert 1 year to seconds.
2. Convert 20 feet/second to miles/hour.
3. Convert 5 euros/kilogram to dollars/pound.
4. Convert 1 mile^2 to acres.
5. Convert 3 gallons to liters.
6. Convert 1 meter^3 to board-feet.
7. Convert 47.1234° to degrees-minutes-seconds form. (Do the arithmetic here.)

14.3 Solutions to Exercises

1.

$$1 \text{ year} \times \frac{365 \text{ days}}{1 \text{ year}} \times \frac{24 \text{ hours}}{1 \text{ day}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} \times \frac{60 \text{ seconds}}{1 \text{ minute}}.$$

2.

$$20 \frac{\text{feet}}{\text{second}} \times \frac{60 \text{ seconds}}{1 \text{ minute}} \times \frac{60 \text{ minutes}}{1 \text{ hour}} \times \frac{1 \text{ mile}}{5280 \text{ feet}}.$$

3.

$$5 \frac{\text{euro}}{\text{kilogram}} \times \frac{1 \text{ kilogram}}{2.2 \text{ pounds}} \times \frac{1 \text{ dollar}}{0.92 \text{ euros}}.$$

4.

$$1 \text{ mile}^2 \times \left(\frac{5280 \text{ feet}}{\text{mile}}\right)^2 \times \left(\frac{1 \text{ yard}}{3 \text{ feet}}\right)^2 \times \frac{1 \text{ chain}}{22 \text{ yards}} \times \frac{1 \text{ furlong}}{220 \text{ yards}} \times \frac{1 \text{ acre}}{1 \text{ chain} \cdot \text{furlong}}.$$

5.

$$3 \text{ gallons} \times \frac{1 \text{ foot}^3}{7.48 \text{ gallons}} \times \left(\frac{12 \text{ inches}}{1 \text{ foot}}\right)^3 \times \left(\frac{1 \text{ meter}}{39.37 \text{ inches}}\right)^3 \times \frac{1000 \text{ liter}}{\text{meter}^3}.$$

6.

$$1 \text{ meter}^3 \times \left(\frac{39.37 \text{ inches}}{1 \text{ meter}}\right)^3 \times \frac{1 \text{ board-foot}}{1 \text{ foot}^2 \cdot \text{inch}} \times \left(\frac{1 \text{ foot}}{12 \text{ inches}}\right)^2.$$

7.

$$\begin{aligned} 47.1234^\circ &= 47^\circ + .1234^\circ = 47^\circ + .1234^\circ \times \frac{60 \text{ minutes}}{1 \text{ degree}} = 47^\circ\ 7.404' \\ &= 47^\circ\ 7' + .404' = 47^\circ\ 7' + .404' \times \frac{60 \text{ seconds}}{1 \text{ minute}} = 47^\circ\ 7'\ 24.24''. \end{aligned}$$