

Trapezoidal Numbers

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July 28, 2008

Abstract

This exercise can be used for middle school students and older. Working on the problem will force the students to do a lot of arithmetic, and they will also learn a (perhaps new) way to understand the ideas behind adding the numbers in an arithmetic sequence.

This document is meant for the teacher and contains pedagogical advice as well as the results.

The original idea and presentation of this topic was done by Paul Zeitz. Tom Davis is the faithful scribe who took notes and wrote up the lesson.

1 The Problem

The concept of a triangular number is fairly simple and well-known. A number is a triangular number if it counts the number of, say, dots, arranged in a triangular pattern. In such a triangle, there is one dot in the first row, two in the second, three in the third, and so forth, so the mathematical notation for a triangular number would be something like this:

$$1 + 2 + 3 + \cdots + n.$$

The number n in the equation above can, of course, be 1, so the first few triangular numbers are:

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \dots$$

Figure 1 shows how the first five triangular numbers can be illustrated by triangles of dots.



Figure 1: Triangular Numbers

2 Initial Analysis

To remind the students (or tell them for the first time) what a triangular number is, draw some triangles of dots like those in Figure 1 and count them. Eventually get them to work out a list like the one above of the first few such numbers. If you want, you can discuss with the students whether the number zero should also be considered to be a triangular number. There's no "official" answer to this, and there are pros and cons to either decision. To make the explanations in this document run most smoothly, it is a good idea to include zero in your list of triangular numbers and we shall do that here.

A trapezoidal number is simply a number that can be drawn using a series of dots that forms a trapezoid (that has at least two rows of dots, or otherwise any straight line of dots would be a trapezoid that is one unit thick). You can draw a few of these as examples as in Figure 2, and try to get the kids to see that any trapezoidal number is simply the difference of two triangular numbers such that the length of the bases of those triangles differ by at least 2.



Figure 2: Trapezoidal Numbers

Tell them also that we would also like to consider the triangular numbers with base at least 2 to be trapezoidal numbers. If you like, and especially if you have decided that zero is a triangular number, they can be considered to be the zero-sized triangle subtracted from a triangle with base at least 2. We will not consider the triangular number 1 to be trapezoidal, since it only has a single layer.

Some groups of students already know a formula for adding the numbers from 1 to n :

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

but it is unlikely that they really have a deep understanding of what it means and where it comes from. But if they do, there's an obvious description of a trapezoidal number:

$$\frac{n(n+1)}{2} - \frac{m(m+1)}{2}, \text{ where } n > m + 1. \quad (1)$$

The condition at the end guarantees that $n > m$, and that the difference in the lengths of the bases of the triangles is at least 2.

If the students do know not know the formulas, do not tell them. If they *do* know the formulas and appear to be satisfied that Equation 1, ask them something like:

“OK, is 1328997488382 a trapezoidal number?” (You can pick any huge multi-digit even number as an example. Use an even one since as we will see shortly, it is fairly obvious that every odd number is trapezoidal.)

Setting the equation with two unknowns n and m equal to your huge number and trying to a solution in m and n that satisfies the conditions should convince them that although Equation 1 *does* characterize the trapezoidal numbers, it is not a particularly useful characterization. We will find one that is much better.

Check the first numbers from 1 to, say, 20 so that you will have a list of triangular and non-triangular numbers against which you can test conjectures. The following form works fine for such a table:

trapezoidal:	3	5	6	7	9	10	11	12	13	14	15	17
non-trapezoidal:	1	2		4		8						16

As you are making this table, the students may come up with some observations. One simple one is that it is easy to make a trapezoid from a given odd number (other than 1, of course). That's because you can divide it almost in half, and one of the “almost halves” will contain one more dot than the other so you can form a trapezoid that is two units high.

Some students may not even know what odd and even numbers are, except for the idea that even numbers end with the digit 0, 2, 4, 6 or 8, and that all the others are odd. Point out to them that even numbers

are divisible by 2, meaning that if you divide them by 2, it goes in exactly and there's no remainder. Obviously, odd numbers are the others: the numbers which, when divided by 2, yield a remainder of 1. Since odd numbers yield a remainder of 1 when divided by 2, this means that they can be written in the form $2k + 1$, where k is a whole number. But $2k + 1 = k + (k + 1)$: two adjacent numbers so that one is exactly one larger than the other, and those can obviously be formed into a two-layer trapezoid.

Another thing that the students may notice is that sometimes many trapezoids can be formed that have the same number of dots. For example, $2 + 3 + 4 = 4 + 5 = 9$, so the number 9 is not only trapezoidal, but can be represented by two different trapezoids. If the students are interested, you could try to find numbers that are representable as trapezoids in a lot of different ways.

Finally, the students should discover that certain numbers, namely 1, 2, 4, 8 and 16 in their list so far, cannot be formed into trapezoids. Ask the students why they are sure that this is the case. How would you prove it?

For small numbers like 1 or 2 it is easy to obtain a proof that they are not trapezoidal, but for a number like 16, how are they sure they've tried all the possibilities. (One reasonable way to prove it is to try to construct trapezoids starting with 1, 2, 3, ... dots, and see if a trapezoid can be formed. Working with 16 as an example, $1 + 2 + 3 + 4 + 5 = 15$, and $1 + 2 + 3 + 4 + 5 + 6 > 16$, so if the top row has one dot in it, no trapezoids can be formed. If 2 dots are on top, $2 + 3 + 4 + 5 = 14$ and $2 + 3 + 4 + 5 + 6 > 16$. Similarly, if 3 is on top, it doesn't work and so on. How far do they have to check? Are there any shortcuts to forming such a proof?

And of course with a list that goes to 20 where all the non-trapezoidal numbers seem to be powers of 2 may lead the students to make that conjecture. If they can do that, now they've got a target for the big result they're trying to prove. If they don't, try to lead them to look for patterns in the trapezoidal and non-trapezoidal properties, and this one should not be too hard to see.

3 Adding Adjacent Numbers

By this point, the students have been getting a lot of practice with their arithmetic, particularly if they're trying to test larger numbers to see if they are trapezoidal. Point out that it would be nice to have an easier way to add lists of consecutive numbers.

Make the following key observation (or better yet, see if you can somehow get a student to make it): If you are trying to add a list of five numbers like, $4 + 5 + 6 + 7 + 8$, notice that this list has a middle number, namely 6, and that there are the same number of numbers above and below the 6. If you pair up 5 and 7 (the numbers just above and below the center), since one is larger by 1 and the other is smaller by 1, their *average* is the same as the number in the middle, namely 6 in this case. The numbers that are 2 above and 2 below the center can similarly be paired, and *their* average will also be 6.

From this, convince the students that the average of all the numbers in the list is 6 since the averages of the subsets $\{6\}$, $\{5, 7\}$ and $\{4, 8\}$ are all 6. Thus the sum of all the numbers in the list is the same as the average (which is 6) multiplied by the total number of entries (which is 5), yielding a sum of $5 \times 6 = 30$. Of course it's easy to check that $4 + 5 + 6 + 7 + 8 = 30$.

Note that, depending on what your students know, this may lead to a discussion of what "average" means: basically, that the sum of a set of numbers is the same as the average of the numbers multiplied by the number of numbers.

So for the sum of an odd number of consecutive numbers you can calculate the sum by taking the middle number and multiplying it by the number of numbers in the sum. What happens with the sum of an even number of consecutive numbers? The problem is that there is no number in the middle. For the sum $5 + 6 + 7 + 8$, the “middle” would be between 6 and 7. See if you can get the students to come up with the idea that 6.5 is the true middle. It’s easy to check that it works as before:

$$5 + 6 + 7 + 8 = 26 = 6.5 \times 4.$$

You can also use the same reasoning that you did with odd lists. In this example, if 6.5 is the average, then 6 and 7 are the same distance above and below it (namely one-half), so the average of 6 and 7 is also 6.5. Similarly, 5 and 8 are 1.5 units below and above the 6.5 average, respectively, so their average is also 6.5, et cetera.

Thus we have a method for finding the sum of any sequence of consecutive integers: find the middle number, and if there is no middle number, use the number half-way between the two numbers in the middle, and multiply that average by the number of numbers in the sum.

You may want to try a few examples with the students to make sure they believe it.

4 Back to the Main Problem

So how can we use this observation that equates the sum of a sequence of consecutive numbers with the result of multiplying the “middle number” by the number of numbers, where “middle” gets the quotation marks because it might be the average of the two numbers in the middle of the list?

Let the students look around for clues, at least for a while, but if they’re not making progress, maybe you can ask them to see if they can show that 42 is a trapezoidal number, given the hint that $42 = 6 \times 7$. If that hint doesn’t work, add that maybe they could make a list of 7 numbers which would behave particularly well according to the previous discovery that a list of 7 numbers will have a middle number.

Hopefully that will be enough, and you will not have to tell them that they can get the 42 with a trapezoidal list of seven numbers where the average, which is to say the middle number, is 6. From that, you can just construct the sequence by putting a 6 in the center and extending it equally to the left and right to make a list of seven:

$$3 + 4 + 5 + 6 + 7 + 8 + 9 = 42,$$

so 42 is a trapezoidal number.

At this point the students should see what is going on, but if not, do the same thing to make a trapezoid containing $70 = 7 \times 10$ dots.

At some point, the students should see what’s going on, and they’ll think they understand. You can show them that they are truly on the right track when you ask them now to show that perfect powers of 2 *cannot* be trapezoidal numbers. The only factors of powers of 2 are even, so every time you try to use the procedure above, you will have a list with an even number of entries, so the average will be a half-integer and thus not suitable for making a trapezoid with that number of dots. You might wish to go through a simple example, like 16, which can be factored either as 2×8 or 4×4 . In every case the sums will have fractional numbers in the centers.

5 One Final Problem

By this time, many of the students will probably be perfectly happy that they have a complete proof of their conjecture that the only non-trapezoidal numbers are the perfect powers of two. If that's the case, have them consider the number $44 = 4 \times 11$. At first they'll say, "All we need is a list of 11 numbers whose middle number is 4". So ask them to do it, and even if they do it "correctly", they'll obtain a list of the following 11 numbers:

$$-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

These numbers do add to make 44, but they don't form a pattern that can easily be visualized as a trapezoidal number. Don't show an answer right away, since maybe the students can figure out how to take care of situations like this. There are various ways to fix the proof, but probably the easiest is to note that if the sum "goes past zero", it's not really a problem. In this case, the -1 and the 1 taken together will have no effect, and neither will the zero, so we have:

$$44 = (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9.$$

The series on the right is a perfectly good representation of 44 as a triangular number.

In the discussion above, there is nothing special about the number 44. I picked it as an example because it's probably larger than any of the examples which were worked out in the class as data to search for a pattern, and it has a form such that the only odd factor is more than twice as big as the remaining even factor. Some easy other examples can be constructed by multiplying 2 or 4 or 8, say, by prime numbers that are more than double the size of the even factor. So you could use 2×17 or 4×13 or 8×29 , for example.

There are other directions to explore as well. Numbers that have complex factorizations can be represented as trapezoidal numbers in lots of ways. The number $54 = 2 \times 3^3$, for example, can be written as 2×27 or 6×9 or 18×3 . Thus (using the techniques above), we can write 54 as 27 numbers with an average of 2, 9 numbers having an average of 6, or 3 numbers having an average of 18. Is that a complete list? Why or why not?