

Conway's Rational Tangles

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1 Introduction

I learned about this “trick” in a lecture by John Conway a number of years ago. He calls it “Rational Tangles” and there is plenty of information about it on the internet. Since then I have used it myself in classrooms of students of middle school age and older. The underlying mathematics is very interesting, but it is not necessary that the students understand the mathematics to get a lot out of the trick. In fact, some of the mathematics I do not understand.

This document is intended for teachers and includes some pedagogical advice on how to use Conway's trick to teach the students something about mathematics.

The idea is that we can associate a number with a tangle of two ropes and that by performing a sequence of two simple operations, we can untangle the ropes in a straightforward way.

Often the best way to get students to practice rote mathematics is to give them a problem that is intrinsically very interesting, but whose solution requires repeated calculations of the sort that you are trying to get them to practice. To do these rope trick calculations, the students will need to practice arithmetic with positive and negative fractions. There is also an opportunity for advanced students to look at far more interesting mathematics.

2 Getting Started

To demonstrate the trick, you need four students and two lengths of rope that are about 10 feet long. Heavier rope is better because it is easier to see the knot structure and it is harder to accidentally pull into tight knots that are difficult to work with. If the ropes are of two different colors, the tangle structure is even easier to see. Bring a few plastic shopping bags.

Get four volunteers to stand at the corners of a rectangle at the front of the class with each student holding one end of a rope. In the initial configuration, the two ropes are parallel to each other and parallel to the front row of seats in the classroom. In Figure 1, the top pair of parallel lines represents the two ropes, and the small circles at the ends with the letters “A”, “B”, “C” and “D” represent the four students. If you imagine that you are looking down on the students from the ceiling, the rest of the class is seated above the entire figure on the page. The four students at the corners face the rest of the class.

Make sure that each student has a solid grip on the rope, perhaps wrapping the end once around their hand so that it is not accidentally dropped. During the trick, no student should ever let go of his or her end of the rope. Don't let the kids start jerking on the rope, since if one end comes loose, it is very easy to lose track

of exactly how the ropes were tangled, and if this occurs, the trick will fail, and the students will lose their interest rapidly. Also, although the trick works for arbitrarily complex tangles, be sure to work with simple ones at first since there is much less chance of an error.

You can explain to the kids that they are going to do something like a square dance where the four students perform one of three “figures”. Also explain that the initial configuration with the parallel, untangled ropes will be assigned the number zero, and that each figure will affect that number in a fixed way. Also, tell the rest of the kids in the class to pay attention, since you’ll swap out sets of kids from time to time so that many more of them can be part of the action.

The only thing that matters is the configuration of the ropes: which student is in which position does not affect the number assigned to a particular tangle.

3 The Three Basic Figures

Conway calls the two main figures “Twist ’em up” and “Turn ’em around”. The unfortunate thing about this choice is that they both begin with the letter “T”. If you’re trying to analyze the results of various sequences, these names do not provide an easy shorthand. Here I will use “Twist” and “Rotate”, since then you can write something like “*TTRTR*” to indicate that sequence of 5 figures in the dance (in this case, two twists, followed by a rotate, then a twist, and finally, another rotate). In what follows, I will use the names **Twist** and **Rotate**, and “T” and “R” as shorthand, especially when I need to refer to a sequence of moves. (In fact, we will see that when a sequence is repeated, we can also use an exponential notation. For example, the sequence *TTTTRTTTR* can be written using the shorthand T^4RT^3R .)

Later in this article we will get even a little more sloppy and say things like “apply a *TTR*” as a shorthand for “apply a **Twist**, then another **Twist** and a **Rotate**”.

When explaining the move “**Twist**”, make sure that all four students pay attention, since although only two of them perform any particular **Twist**, they may be arranged differently later in the dance and will have to do it when they are in those positions.

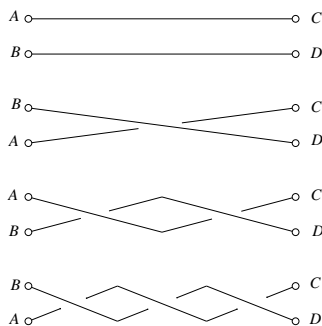


Figure 1: Twisting

To perform the **Twist** figure, the two students on the left (from the point of view of the students) change places, with the student initially in the rear lifting his or her rope and the student in front stepping back under it. As it is labeled in Figure 1, students A and B swap places, where B lifts his or her rope and A steps

under it to the rear. In Figure 1 the results of performing zero, one, two and three of the **Twist** figures (or, in shorthand, performing T , T^2 , T^3 and T^4) from the initial (zero) configuration are shown from top to bottom. Notice that from the points of view of *both* of the students holding the ropes, after starting from zero and performing just **Twist** moves, the ropes will appear to twist away from them in a clockwise direction. Notice that with each **Twist**, the positions of students A and B are swapped. Demonstrate this to the class.

Each time a **Twist** move is made, the number associated with the tangle is increased by 1, so in Figure 1 the four tangles from top to bottom are represented by the numbers 0, 1, 2 and 3. If the tangle's number is a pure integer like this, then the integer represents the number of half-twists in the rope.

Conway's figures for manipulating tangles do not admit an **UnTwist** figure (which would exactly undo a **Twist**), but if there were one, it would be easy to do: the same two people on the left change places, but this time the person in front raises the rope and the person behind steps under it. Such an **UnTwist** figure would subtract 1 from the tangle's associated number. This is a very obvious concept, and if **Twist** and **UnTwist** were the only two legal moves, it's clear that starting from zero, any positive or negative integer could be obtained, and if you knew that number, the ropes could be untangled by performing that number of **UnTwists** or **Twists**, depending on whether the number were positive or negative.

The second figure, **Display**, does nothing to the tangle; it is simply to display the condition of the ropes and tangle to the rest of the class. To do a **Display**, the two people farthest from the class raise their ropes and the two in front lower them so the tangle is displayed in an unobstructed way. Conway usually also required that everyone in the class cheer and clap when a **Display** figure was performed.

To perform the third figure, **Rotate**, each student moves one position clockwise, when viewed from above. In Figure 1 if we began from the top arrangement in the figure, a **Rotate** would move A to C's position, B to A's position, D to B's position and C to D's position. If you were to **Rotate** four times in a row, each student would wind up exactly where they started. Demonstrate to the class that at least when there are only twists in the rope (and in fact it will always be true) that two **Rotate** figures will return the ropes back to where they started, even though the students will be on the opposite sides. Perhaps this can be made clear by reminding the students that the number 3, for example, represents 3 clockwise half-twists of the ropes from the point of view of any of the students. As they turn around, nothing is going to change the clockwise orientation, so after the two pairs on the ends have swapped places, they still see three clockwise half-twists, so rope configuration is unchanged.

This observation indicates that the operation on the number associated with the tangle has to bring it back to where it started if you apply that operation twice. Depending on the sophistication of the class, you might use functional notation as follows:

Let x be the number associated with the current tangle. If we apply a **Twist**, we'll use the function $t(x) = x + 1$ to indicate what a **Twist** does to the current number. At this point, we don't know what the **Rotate** number $r(x)$ does, but we do know two things. Clearly, $r(r(r(r(x)))) = x$, since rotating everyone completely around the square obviously leaves everything completely unchanged, no matter what the tangle. We also know for sure that if the tangle consists only of twists, then $r(r(x)) = x$.

Warning: This functional notation may be confusing, since the functions have to be written in the opposite order that the dance figures are performed. For example, if we start from a tangle whose associated number is x and do a **Twist** followed by a **Rotate**, we've been using the notation " TR " to indicate that: "twist, then rotate". But to figure out what the resulting corresponding number should be, the **Twist** will turn x into $t(x)$ and the **Rotate** function will operate on $t(x)$ to produce $r(t(x))$. It's easy to see how this reversal will always

occur, so that something like “ $TTRTTTTRT = T^2RT^3RT$ ” will convert an initial number x to:

$$t(r(t(t(t(r(t(t(x))))))))).$$

So the bottom line is that unless you’ve got a sophisticated audience, it’s probably a good idea to avoid the functional notation.

At this point we still don’t know exactly how **Rotate** should affect the tangle’s associated number. All we know (or at least suspect) is that applying **Rotate** twice brings us back to where we started. In other words, $r(r(x)) = x$.

For another clue about how **Rotate** should affect the number (or alternatively, the form of $r(x)$), have the students do this: Start from the ropes in a “zero” tangle. Do one **Twist** (so the number is now 1). Next do a **Rotate**. Finally do another **Twist**, and they will find that this brings the ropes back to the untangled state; namely, zero. This means that after the **Rotate**, the number must have been -1 , since adding 1 to it brings us back to a 0 configuration. So **Rotate** changes a 1 to a -1 . (Alternatively, using the functional notations we could write: $r(1) = -1$.)

The class will then probably make the reasonable (but wrong) guess that a **Rotate** figure multiplies the number by -1 . Sometimes they even guess that it adds 2. You can convince them that adding 2 is clearly wrong, since doing it twice should return to the original number and adding 2 twice will add 4 to the original number. The conjecture that **Rotate** multiplies the number by -1 (or functionally, that $r(x) = -x$) makes sense, since multiplying by -1 twice returns to the initial number. But it’s easy to test: Start from 0, do two **Twists** (which will convert the tangle’s number to 2) followed by a **Rotate**. If **Rotate** multiplies by -1 , then the ropes should then be in the -2 state, and two **Twists** should add 2 to the -2 , returning the tangle to the initial state. Try it, and see that this does not happen.

Depending on the sophistication of the class, you can either tell them the answer, or try to lead them to it by considering other operations that turn 1 into -1 but not 2 into -2 , yet when repeated twice bring every number back to itself.

Some classes have students who have a good enough geometric intuition to look at a tangle as simple as that generated by starting at zero and then performing $TTR = T^2R$ (alternatively, producing $r(t(t(n)))$). The sequence that undoes it is $TRTT = TRT^2$, for some reason. At least that’s another data point.

If the class is a little bit sophisticated, here is another nice thing to do. Start with the usual zero configuration and do a single **Rotate**. Now do any number of **Twist** operations. The **Twist** operations have no effect: the ropes remain parallel to each other, but perpendicular to the front row of the class. This means that whatever number $r(0)$ happens to be (let’s call it i , for now), this “number” i has the strange property that $i + 1 = i$. Now there aren’t any normal numbers like that, but maybe one of the kids will come up with the idea that $\infty + 1 = \infty$, so perhaps $i = \infty$, whatever that means.

If that’s the case, we have $r(r(x)) = x$, we have $r(1) = -1$ and we have $r(0) = \infty$ (and obviously, if we rotate the “ ∞ ” configuration we’ll return to zero, so $r(\infty) = 0$). These may provide enough clues for a sophisticated class to determine the correct operation to associate with the **Rotate** figure. (Actually, an even better number to assign might be $-\infty$, but that may require even more sophistication on the part of the students.)

The correct answer is that **Rotate** takes the tangle represented by x and turns it into the tangle represented by $-1/x$. Thus, starting from zero, the sequence TTR leaves a tangle with value $-1/2$. You can check this by starting from zero, doing a TTR (which should leave $-1/2$), then doing a T (yielding a value of $-1/2 + 1 = +1/2$), then an R (yielding -2) and then a TT brings you back to zero. Have the class check

that this works for the examples examined so far, and if they've discovered the ∞ idea, that it makes some sort of sense even then.

See if the kids can figure out how to get a TTT (with value 3) back to zero. It's a little bit complicated and requires 8 steps which are listed below. If the kids have no idea how to proceed, the following clue may be enough to get them going: "We're starting at +3 and trying to get to zero. If we do a **Twist**, we'll just be at +4, which is even farther from our goal." This should lead the kids to think that a **Rotate** is the best way to proceed. After they've done the **Rotate** command, if they're still wondering what to do, say, "If you do another **Rotate**, it'll just undo the effects of your previous **Rotate**". Thus it may be clear that the only way to make progress toward zero from +3 is to begin with RT . In any case, let them work on it for a while, but here is the fastest sequence that returns +3 to zero:

$$3 \xrightarrow{R} -\frac{1}{3} \xrightarrow{T} \frac{2}{3} \xrightarrow{R} -\frac{3}{2} \xrightarrow{TT} \frac{1}{2} \xrightarrow{R} -2 \xrightarrow{TT} 0.$$

4 Getting to Zero

At this point you can begin to consider how an arbitrary tangling of the rope using the **Twist** and **Rotate** commands can be converted back to the untangled state again using only the **Twist** and **Rotate** commands. You might begin by having the kids suggest moves that tangle the rope a little bit: perhaps seven or eight steps, but carefully keeping track of the numbers. For now, make sure they start with at least two **Twists** and mix in both **Twists** and **Rotates** after that.

If they try to do two **Rotates** in a row, point out that although this is perfectly legal, the second **Rotate** undoes the first, so the two moves taken together achieve nothing.

What you can do is collect move suggestions from different kids, keeping track of the tangle's number until it is "suitably" complex. Also, as your first example, stop them after a command has left the number positive. For example, the sequence $TRRTTTRT$, starting from zero, leaves a tangle with associated number $3/5$. This is a nice number since it doesn't have numerator or denominator that's too big, but it is complicated enough to be interesting. We'll use this example in what follows.

Tell the kids that their goal is to get the number $3/5$ down to zero using only **Rotate** and **Twist** commands. As a first hint, tell them that **Twist** will add 1 which will take the number even farther away from zero, so to make progress, the only possible command is **Rotate**. We now have $-5/3$.

Point out now that another **Rotate** will just undo the one they did, so the only reasonable next step is a **Twist**, yielding $-2/3$. At this point a **Rotate** will not put you back where you started, but it would yield a positive number, and that can't be good, since another **Rotate** is useless, and one or more **Twist** commands would take the number away from zero. Thus from $-2/3$, the only reasonable move is another **Twist**, yielding $1/3$.

Repeating the arguments above, we clearly need a **Rotate**, taking us to -3 , and then three **Twists** get back to zero. Go ahead and do this with the ropes and verify that indeed it does untangle the mess.

If you haven't done it already, this is a good time to swap in a new set of four students.

Now for the best part: make another tangle, a bit more involved than the last one, and once it's created, put the tangle into a bag as follows. Take a plastic bag and make two small holes in the corners opposite the opening. Take the ropes, one at a time, from the two kids on the left and feed them through the holes and back to the kid. Pull the bag opening over the tangle and tie the whole thing shut so that the tangle is completely enclosed in the bag.

Finally, carefully apply the steps that undo the tangle and when you're done, there will be a horrible snarl of ropes and plastic, which, if you've made no mistakes, should be equivalent to zero. To prove it, tear the plastic bag into pieces to extract it from the tangle, and then with a few tugs, the entire mess will appear to magically untangle itself!

It's sometimes fun to do this more than once, so bring more than one plastic bag to the class.

5 “Bad” Tangles

The following tangles are perfectly ok in a mathematical sense, but untangling them is a long process, and can be quite error-prone. For that reason, unless you've got masochistic tendencies, avoid tangles with numbers like $-1/n$, where n is a large integer. They are easy to produce: suppose you start from zero and do *TTTTTTTR*: eight **Twist** figures followed by a **Rotate**. This will produce the number $-1/8$. It's a good exercise to try to untangle it using our method to see what happens. Here's what happens, shown as a series of steps:

Start	Operation	Result
$-1/8$	TRT	$-1/7$
$-1/7$	TRT	$-1/6$
$-1/6$	TRT	$-1/5$
$-1/5$	TRT	$-1/4$
$-1/4$	TRT	$-1/3$
$-1/3$	TRT	$-1/2$
$-1/2$	TRT	-1
-1	T	0

It requires 22 moves to return $-1/8$ to zero. Not only that, but the arithmetic is pretty boring: go through the details for a couple of the rows above to see what happens.

6 Discussion Topics

Here are a few ideas that may lead to interesting class discussions:

6.1 Infinity as a Tangle Number

Try starting with zero and do a single **Rotate**. This yields the nonsense value $-1/0$, but it's not a nonsense tangle. Another **Rotate** will bring it back to zero, and in fact, it sort of behaves like “infinity” in the sense that a **Twist** (try it) leaves it exactly the same. This sort of corresponds to the idea that adding 1 to ∞ leaves it unchanged. You may have discussed this earlier, depending on the sophistication of the class.

6.2 Proof of Convergence to Zero

Can you prove that the scheme outlined above will always eventually grind any initial fraction down to zero? Go through a few examples and see what is happening. Here is an example starting from $-5/17$:

$$-\frac{5}{17} \xrightarrow{T} \frac{12}{17} \xrightarrow{R} -\frac{17}{12} \xrightarrow{TT} \frac{7}{12} \xrightarrow{R} -\frac{12}{7} \xrightarrow{TT} \frac{2}{7} \xrightarrow{R} -\frac{7}{2} \xrightarrow{TTTT} \frac{1}{2} \xrightarrow{R} -\frac{2}{1} \xrightarrow{TT} 0.$$

Note that after each **Rotate** command, the resulting negative fraction has a smaller denominator. Why is this? If the denominators always eventually get smaller, they must eventually get to 1. But when a denominator is 1 the fraction will be a negative integer, and if that integer happens to be $-n$, we know that n **Twist** commands (each adding 1) will reduce it to zero.

6.3 Relationship to the Greatest Common Divisor

If the students are a bit advanced, you can point out that the process of reducing the fraction down to zero is almost exactly the same as finding the greatest common divisor (the *GCD*) of the numerator and denominator. Since we begin with a fraction reduced to lowest terms, this will always get us down to 1 as the *GCD*.

Euclid's algorithm for calculating the *GCD* of two numbers works as follows. If the two numbers are m and n , and we suppose that $m > n$, we can write: $m = kn + l$, where $k \geq 1$ is an integer and $|l| < n$. Any number that divides m and n must divide l in the equation above, so we can conclude that $GCD(m, n) = GCD(n, l)$. The numbers in the right hand side are reduced, and the process can be repeated until one is a multiple of the other.

Here is an example: find the *GCD* of 4004 and 700:

$$\begin{aligned} 4004 &= 700 \times 5 + 504 \\ 700 &= 504 \times 1 + 196 \\ 504 &= 196 \times 2 + 112 \\ 196 &= 112 \times 1 + 84 \\ 112 &= 84 \times 1 + 28 \\ 84 &= 28 \times 3. \end{aligned}$$

The *GCD* of 4004 and 700 must divide 504 from the first line, so $GCD(4004, 700) = GCD(700, 504)$. The same process can be continued to obtain:

$$\begin{aligned} GCD(4004, 700) &= GCD(700, 504) = GCD(504, 196) \\ &= GCD(196, 112) = GCD(112, 84) = GCD(84, 28). \end{aligned}$$

But 84 is an exact multiple of 28, so $GCD(84, 28) = 28$, and we can therefore conclude that

$$GCD(4004, 700) = 28$$

as well.

Note that there is no requirement that the numbers on the right hand sides of the sequence of reductions be positive. All that we require for convergence is that they be smaller in absolute value than the smaller of the

two values for which you are trying to obtain the GCD . Also note that division can be achieved by repeated subtraction, and if you simply check to see if the subtraction yields a non-positive number, you know you have gone far enough. It's sort of like backing up your car until you hear breaking glass, but it works!

With all that in mind, let's find the GCD of 5 and 17 using this totally crude method:

$$\begin{aligned} 5 &= 17 \times 1 - 12 \\ 17 &= 12 \times 1 + 5 = 12 \times 2 - 7 \\ 12 &= 7 \times 1 + 5 = 7 \times 2 - 2 \\ 7 &= 2 \times 1 + 5 = 2 \times 2 + 3 = 2 \times 3 + 1 = 2 \times 4 - 1 \\ 2 &= 1 \times 1 + 1 = 1 \times 2 + 0. \end{aligned}$$

Note the similarity of this method to the one we used to obtain $GCD(4004, 700)$ above. But this time, rather than doing a division, we do repeated subtractions until the remainder is zero or negative. Then we use the (positive value of) the remainder in the next step. We finally discover that 1 divides 2 evenly, so $GCD(5, 17) = 1$. Now compare this sequence to the one that reduces the tangle value $-5/17$ to zero at the beginning of this section. You will see that the calculations are virtually identical.

6.4 What Tangle Numbers Are Possible?

Is it possible to start from zero and get to any (positive or negative) fraction? Have the students mess around for a while and see what fractions they can come up with. Also, set goals, such as, "Can you start from zero and get to -3 ?" If there is no progress, here is a giant hint:

$$\begin{aligned} \frac{3}{1} &\xrightarrow{R} -\frac{1}{3} \xrightarrow{T} \frac{2}{3} \xrightarrow{R} -\frac{3}{2} \xrightarrow{TT} \frac{1}{2} \xrightarrow{R} -\frac{2}{1} \xrightarrow{TT} 0 \\ 0 &\xrightarrow{TT} \frac{2}{1} \xrightarrow{R} -\frac{1}{2} \xrightarrow{TT} \frac{3}{2} \xrightarrow{R} -\frac{2}{3} \xrightarrow{T} \frac{1}{3} \xrightarrow{R} -\frac{3}{1}. \end{aligned}$$

If we start from 3 and work our way to zero using our standard methods, the sequence $RTTRTTRTT$ does the trick. But now note that if we start from zero and use the reverse of the sequence above, namely: $TTRTTRTR$, we get to -3 . Also, note that at every stage in the sequence, the same fractions are generated, except that they have opposite signs.²

6.5 Minimum Steps from 0

From the previous section, we know that we can get to any fraction i/j by using our algorithm to grind $-i/j$ to zero, and then reversing the order of the figures. Is this the minimum number of steps?

The tables below show the minimum number of steps to get to positive and negative fractions between $1/1$ and $7/7$ (the second shows the steps to get fractions between $-1/1$ and $-7/7$). We use exponents to reduce the size of the strings in the sense that we might express the sequence $TTRTTRTTRTT$ as $T(TRT)^2T^2$. An X in a slot indicates that the fraction represented by that slot is not reduced to lowest terms. So for example, to look up the shortest sequence that will get you from zero to $-5/7$, we use the second table below (since the fraction is negative). We look in the column headed by -5 and the row headed by 7 and find the following:

$T(TTR)^3$, which would expand to $TTTRTTTRTTR$, and it is easy to verify that this, in fact, will generate $-5/7$. Check some other examples.

Similarly, if you look in the column headed by 6 and the row headed by 4, you find an X , since $6/4$ is not reduced to lowest terms: you should have been looking for $3/2$ in column 3, row 2.

Table 1: Positive fractions:

	1	2	3	4	5	6	7
1	T	T^2	T^3	T^4	T^5	T^6	T^7
2	T^2RT	X	T^2RT^2	X	T^2RT^3	X	T^2RT^4
3	$T(TRT)^2$	T^3RT	X	$T(TRT)^2T$	T^3RT^2	X	$T(TRT)^2T^2$
4	$T(TRT)^3$	X	T^4RT	X	$T(TRT)^3T$	X	T^4RT^2
5	$T(TRT)^4$	$T^2(TRT)^2$	T^2RT^3RT	T^5RT	X	$T(TRT)^4T$	$T^2(TRT)^2T$
6	$T(TRT)^5$	X	X	X	T^6RT	X	$T(TRT)^5T$
7	$T(TRT)^6$	$T^2(TRT)^3$	$T^3(TRT)^2$	$T(TRT)^2T^2RT$	T^2RT^4RT	T^7RT	X

Table 2: Negative fractions:

	-1	-2	-3	-4	-5	-6	-7
1	TR	$T(TRT)R$	$T(TRT)^2R$	$T(TRT)^3R$	$T(TRT)^4R$	$T(TRT)^5R$	$T(TRT)^6R$
2	T^2R	X	$T^2(TRT)R$	X	$T^2(TRT)^2R$	X	$T^2(TRT)^3R$
3	T^3R	T^2RT^2R	X	T^4RTR	T^2RT^3RTR	X	T^4RT^2RTR
4	T^4R	X	$(T^2R)^3$	X	T^5RTR	X	$(T^2R)^2T^3RTR$
5	T^5R	T^2RT^3R	T^3RT^2R	$T(TRT)^3TR$	X	T^6RTR	T^2RT^4RTR
6	T^6R	X	X	X	$(T^2R)^5$	X	T^7RTR
7	T^7R	T^2RT^4R	$(T^2R)^2T^3R$	T^4RT^2R	$T(TTR)^3$	$(TTR)^6$	X

There are some obvious patterns here, and an interesting exercise is both to look at the patterns and then to try to prove them. Here is a list of some of the obvious ones; perhaps there are others.

$$\begin{aligned}
T^n &: 0 \longrightarrow n \\
T(TRT)^n &: 0 \longrightarrow 1/(n+1) \\
T^2RT^n &: 0 \longrightarrow (2n-1)/2 \\
T^2(TRT)^n &: 0 \longrightarrow 2/(2n+1) \\
T(TRT)^nR &: 0 \longrightarrow -(n+1)
\end{aligned}$$

On the following page is a list of fractions that can be obtained, starting from zero, by applying various sequences of T and R . The data on that page may also be useful to generate conjectures about sequence patterns and the fractions resulting from them.

Some sequences do not yield patterns that are obvious at first. For example, consider the sequence: T^2RT , T^3RT^2 , T^4RT^3 , \dots , in other words, what does $T^{n+1}RT^n$ represent? The first few values are $1/2$, $5/3$, $11/4$, $19/5$, $29/6$. The denominators go up by 1 each time and the numerators by 4, 6, 8, 10. A little fooling around will yield the formula $(n(n+1)-1)/(n+1)$, for $n > 0$. Following this idea can lead to an investigation of how to find formulas to represent the numbers in some series.

Here is a table listing the resulting fraction from various sequences of T and R :

R	∞	T^5RT^2R	$-5/9$	T^8RTR	$-8/7$	T^7RT^2RT	$6/13$
T	$1/1$	T^4RT^4	$15/4$	T^7RT^3	$20/7$	T^6RT^5	$29/6$
T^2	$2/1$	T^4RT^3R	$-4/11$	T^7RT^2R	$-7/13$	T^6RT^4R	$-6/23$
TR	$-1/1$	T^4RT^2RT	$3/7$	T^6RT^4	$23/6$	T^6RT^3RT	$11/17$
T^3	$3/1$	T^3RT^5	$14/3$	T^6RT^3R	$-6/17$	$T^6RT^2RT^2$	$16/11$
T^2R	$-1/2$	T^3RT^4R	$-3/11$	T^6RT^2RT	$5/11$	T^6RT^2RTR	$-11/5$
T^4	$4/1$	T^3RT^3RT	$5/8$	T^5RT^5	$24/5$	T^5RT^6	$29/5$
T^3R	$-1/3$	$T^3RT^2RT^2$	$7/5$	T^5RT^4R	$-5/19$	T^5RT^5R	$-5/24$
T^2RT	$1/2$	T^3RT^2RTR	$-5/2$	T^5RT^3RT	$9/14$	T^5RT^4RT	$14/19$
T^5	$5/1$	T^2RT^6	$11/2$	$T^5RT^2RT^2$	$13/9$	$T^5RT^3RT^2$	$23/14$
T^4R	$-1/4$	T^2RT^5R	$-2/9$	T^5RT^2RTR	$-9/4$	T^5RT^3RTR	$-14/9$
T^3RT	$2/3$	T^2RT^4RT	$5/7$	T^4RT^6	$23/4$	$T^5RT^2RT^3$	$22/9$
T^2RT^2	$3/2$	$T^2RT^3RT^2$	$8/5$	T^4RT^5R	$-4/19$	$T^5RT^2RT^2R$	$-9/13$
T^2RTR	$-2/1$	T^2RT^3RTR	$-5/3$	T^4RT^4RT	$11/15$	T^4RT^7	$27/4$
T^6	$6/1$	$T^2RT^2RT^3$	$7/3$	$T^4RT^3RT^2$	$18/11$	T^4RT^6R	$-4/23$
T^5R	$-1/5$	$T^2RT^2RT^2R$	$-3/4$	T^4RT^3RTR	$-11/7$	T^4RT^5RT	$15/19$
T^4RT	$3/4$	T^{10}	$10/1$	$T^4RT^2RT^3$	$17/7$	$T^4RT^4RT^2$	$26/15$
T^3RT^2	$5/3$	T^9R	$-1/9$	$T^4RT^2RT^2R$	$-7/10$	T^4RT^4RTR	$-15/11$
T^3RTR	$-3/2$	T^8RT	$7/8$	T^3RT^7	$20/3$	$T^4RT^3RT^3$	$29/11$
T^2RT^3	$5/2$	T^7RT^2	$13/7$	T^3RT^6R	$-3/17$	$T^4RT^3RT^2R$	$-11/18$
T^2RT^2R	$-2/3$	T^7RTR	$-7/6$	T^3RT^5RT	$11/14$	$T^4RT^2RT^4$	$24/7$
T^7	$7/1$	T^6RT^3	$17/6$	$T^3RT^4RT^2$	$19/11$	$T^4RT^2RT^3R$	$-7/17$
T^6R	$-1/6$	T^6RT^2R	$-6/11$	T^3RT^4RTR	$-11/8$	$T^4RT^2RT^2RT$	$3/10$
T^5RT	$4/5$	T^5RT^4	$19/5$	$T^3RT^3RT^3$	$21/8$	T^3RT^8	$23/3$
T^4RT^2	$7/4$	T^5RT^3R	$-5/14$	$T^3RT^3RT^2R$	$-8/13$	T^3RT^7R	$-3/20$
T^4RTR	$-4/3$	T^5RT^2RT	$4/9$	$T^3RT^2RT^4$	$17/5$	T^3RT^6RT	$14/17$
T^3RT^3	$8/3$	T^4RT^5	$19/4$	$T^3RT^2RT^3R$	$-5/12$	$T^3RT^5RT^2$	$25/14$
T^3RT^2R	$-3/5$	T^4RT^4R	$-4/15$	$T^3RT^2RT^2RT$	$2/7$	T^3RT^5RTR	$-14/11$
T^2RT^4	$7/2$	T^4RT^3RT	$7/11$	T^2RT^8	$15/2$	$T^3RT^4RT^3$	$30/11$
T^2RT^3R	$-2/5$	$T^4RT^2RT^2$	$10/7$	T^2RT^7R	$-2/13$	$T^3RT^4RT^2R$	$-11/19$
T^2RT^2RT	$1/3$	T^4RT^2RTR	$-7/3$	T^2RT^6RT	$9/11$	$T^3RT^3RT^4$	$29/8$
T^8	$8/1$	T^3RT^6	$17/3$	$T^2RT^5RT^2$	$16/9$	$T^3RT^3RT^3R$	$-8/21$
T^7R	$-1/7$	T^3RT^5R	$-3/14$	T^2RT^5RTR	$-9/7$	$T^3RT^3RT^2RT$	$5/13$
T^6RT	$5/6$	T^3RT^4RT	$8/11$	$T^2RT^4RT^3$	$19/7$	$T^3RT^2RT^5$	$22/5$
T^5RT^2	$9/5$	$T^3RT^3RT^2$	$13/8$	$T^2RT^4RT^2R$	$-7/12$	$T^3RT^2RT^4R$	$-5/17$
T^5RTR	$-5/4$	T^3RT^3RTR	$-8/5$	$T^2RT^3RT^4$	$18/5$	$T^3RT^2RT^3RT$	$7/12$
T^4RT^3	$11/4$	$T^3RT^2RT^3$	$12/5$	$T^2RT^3RT^3R$	$-5/13$	$T^3RT^2RT^2RT^2$	$9/7$
T^4RT^2R	$-4/7$	$T^3RT^2RT^2R$	$-5/7$	$T^2RT^3RT^2RT$	$3/8$	$T^3RT^2RT^2RTR$	$-7/2$
T^3RT^4	$11/3$	T^2RT^7	$13/2$	$T^2RT^2RT^5$	$13/3$	T^2RT^9	$17/2$
T^3RT^3R	$-3/8$	T^2RT^6R	$-2/11$	$T^2RT^2RT^4R$	$-3/10$	T^2RT^8R	$-2/15$
T^3RT^2RT	$2/5$	T^2RT^5RT	$7/9$	$T^2RT^2RT^3RT$	$4/7$	T^2RT^7RT	$11/13$
T^2RT^5	$9/2$	$T^2RT^4RT^2$	$12/7$	$T^2RT^2RT^2RT^2$	$5/4$	$T^2RT^6RT^2$	$20/11$
T^2RT^4R	$-2/7$	T^2RT^4RTR	$-7/5$	$T^2RT^2RT^2RTR$	$-4/1$	T^2RT^6RTR	$-11/9$
T^2RT^3RT	$3/5$	$T^2RT^3RT^3$	$13/5$	T^{12}	$12/1$	$T^2RT^5RT^3$	$25/9$
$T^2RT^2RT^2$	$4/3$	$T^2RT^3RT^2R$	$-5/8$	$T^{11}R$	$-1/11$	$T^2RT^5RT^2R$	$-9/16$
T^2RT^2RTR	$-3/1$	$T^2RT^2RT^4$	$10/3$	$T^{10}RT$	$9/10$	$T^2RT^4RT^4$	$26/7$
T^9	$9/1$	$T^2RT^2RT^3R$	$-3/7$	T^9RT^2	$17/9$	$T^2RT^4RT^3R$	$-7/19$
T^8R	$-1/8$	$T^2RT^2RT^2RT$	$1/4$	T^9RTR	$-9/8$	$T^2RT^4RT^2RT$	$5/12$
T^7RT	$6/7$	T^{11}	$11/1$	T^8RT^3	$23/8$	$T^2RT^3RT^5$	$23/5$
T^6RT^2	$11/6$	$T^{10}R$	$-1/10$	T^8RT^2R	$-8/15$	$T^2RT^3RT^4R$	$-5/18$
T^6RTR	$-6/5$	T^9RT	$8/9$	T^7RT^4	$27/7$	$T^2RT^3RT^3RT$	$8/13$
T^5RT^3	$14/5$	T^8RT^2	$15/8$	T^7RT^3R	$-7/20$	$T^2RT^3RT^2RT^2$	$11/8$