# Visualization and Symmetry
## (Preliminary)

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October 12, 2015

**Abstract**

We will examine visualization and symmetry in a very general way by means of a set of problems. Many topics in mathematics can be made much clearer when symmetric aspects are made clear or when nice alternative visualizations are possible. When this occurs, it helps both the student and the teacher.

There is a large amount of potential classroom material here, and almost any small part of it could be used for an entire class session.

## 1 Introduction to Visualization

**Note:** This document is unfinished, mainly because the solutions are not all yet included. It takes a lot more time to write up solutions than the corresponding problems, so as I have time, this document will become more and more complete. It is certainly useful as it is, but will be even better in the future. — the author.

All of us (including both our students and ourselves) think differently. Some of our brains work well manipulating symbols (algebraic computations, for example) and others of us are better at imagining and mentally manipulating shapes and diagrams (we'll call this geometric manipulation). Often problems can be viewed both algebraically and geometrically, and can be attacked using both methods.

We will use the term "visualization" here in a very general way. Basically, the idea is to try to find different ways to think about each problem since each different view gives us more understanding. The more different ways you have of looking at a problem, the better you will understand it.

**Note:** Some of the exercises below are marked with one or two asterisks: (*) or (**). These indicate problems that may be more difficult or much more difficult, respectively, than the others. Of course these are totally subjective determinations by the author; different people have different talents.

This article is still a work in progress, and not all the solutions are complete. Any problem that has a solution here will include a number, like "Solution: Item 5". This means that item number 5 in Section 7 is a solution (or at least a hint) for that problem.

## 2 Visualization of Algebraic Manipulation Rules

As a first example, we will try to come up with a set of ways that students can think about algebraic concepts in a way that makes them not just a set of somewhat arbitrary rules, but as sensible ideas that are "obviously" true. There's nothing special about the examples below, except that whenever we present a new topic in class, it's good to try to think of ways that the new concept is "natural", based on what the students already know.

1. **The distributive law.** Rather than just the sterile formula that states that given any three real numbers $A$, $B$ and $C$ that:

$$A(B+C) = AB + AC$$

   why not approach it this way:

"The parentheses group things together. Suppose we're thinking about a bunch of married couples, each of which consists of a man and a woman and we could write an 'equation' that looks something like this:

$$\text{couple} = (\text{man} + \text{woman}).$$

The parentheses indicate that the man and woman are grouped together. What would $8$ such married couples look like? Well, it would be $8$ copies of that group:

$$8 \text{ couples} = 8(\text{man} + \text{woman}).$$

But isn't it obvious that this would amount to $8$ men and $8$ women? We're making $8$ copies of the group, so that would be $8$ of everything in the group."

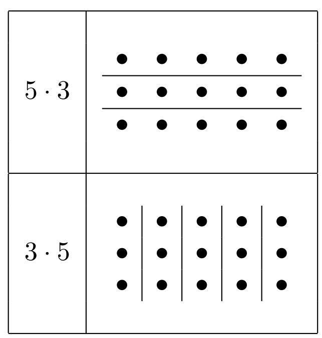
Next you could look at something slightly more complex, like packs that consist of $5$ baseball cards and one piece of chewing gum. What would 3 of those packs look like? Well:

$$3 \text{ packs} = 3(5 \text{ cards} + 1 \text{ gum}).$$

It should be clear what the resulting collection consists of; namely, $3 \cdot 5 = 15$ baseball cards and $3 \cdot 1 = 3$ pieces of gum.

2. **The commutative laws for addition and multiplication.** This can probably be done with pictures that look something like these:

| | |
|---|---|
| $7+3$ | $\bullet\bullet\bullet\bullet\bullet\bullet\bullet+\bullet\bullet\bullet$ |
| $3+7$ | $\bullet\bullet\bullet+\bullet\bullet\bullet\bullet\bullet\bullet\bullet$ |




If we only talk about addition and multiplication, students may think, "Why even mention the commutative law? It's obvious." So look at some operations that are *not* commutative, like subtraction and division. What about exponentiation?

3. **Associative laws.** By looking at the combination of dots as in the examples above, the associative laws or addition and multiplication can be made clear. This is very easy for additon:

$$(2+3)+4 = 2+(3+4)$$

is equivalent to:

$$(\bullet\bullet+\bullet\bullet\bullet)+\bullet\bullet\bullet\bullet = \bullet\bullet+(\bullet\bullet\bullet+\bullet\bullet\bullet\bullet).$$

For multiplication, the product of three numbers can be viewed as 3D blocks of "dots". If we agree that $a \times b \times c$ refers to a block of width $a$, length $b$ and height $c$, then the two groupings that the distributive law declares to be the same just amount to slicing the block in different orders. The same total number of dots remains the same.

As we did with the commutative law, it's a good idea to look at examples of operations that are *not* associative. Again, subtraction and division are good examples. What about exponentiation?

4. **Combining like terms.** When asked to take an expression and "simplify" it, the following expression:

$$2xy + 3xy^3 + z + 2xy^3 + 3z$$

is probably a far more frightening example than:

$$2\text{ dogs} + 3\text{ cats} + 1\text{ bird} + 2\text{ cats} + 3\text{ birds}.$$

If we think of "$xy$" as "dog", "$xy^3$" as "cat" and so on, the two "expressions" above are equivalent.

For students who may try to combine terms like $3x$ and $4xy$ since they share an $x$, you may be able to show them why this won't work because it should be obvious that there's no way to combine 3 "dogs" with $4$ "doghouses": the terms have to be identical before you can sum the constants.

5. (*) Nice examples for the use of the commutative and distributive laws and combination of like terms can be sought in ordinary arithmetic. If the students remember how to add and multiply, these operations can be used as examples. If they're rusty, maybe the laws can help remember the operations:

$$7 \times 368 = 7 \times (3 \times 100 + 6 \times 10 + 8) \ldots$$

Similarly, addition is combining like terms, then carrying is regrouping.

6. (*) Possible idea: could polynomial multiplication (in one variable) be made clear in terms of standard multiplication? For example, if we agree that $x = 10$, then the following are equivalent:

$$123 \times 456 = (x^2 + 2x + 3)(4x^2 + 5x + 6).$$

The only problem here is that polynomial multiplication does nothing about carrying. In a sense, it's a shame that carrying makes ordinary arithmetic multiplication more difficult than polynomial multiplication.

7. Can we come up with others?

# 3 More Visualization Exercises

Mathematics uses both the (symbol-manipulating) left brain and the (visual, geometric) right brain. Both are important, although the emphasis in elementary mathematics courses is generally on "left-brained" activity. Following are some exercises in visualization, some general, and some specifically aimed at problems related to elementary algebra.

What we will do in this section is look at "translations" of algebraic problems into possibly more easily-visualized geometric problems.

## 3.1 Adding Series of Numbers

1. Summing the basic series. We will work out this first example in detail. Following are more examples that can be approached in a similar manner.

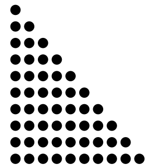
As an example, suppose we need to find the sum:

$$1 + 2 + 3 + ... + 100.$$

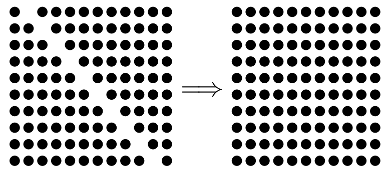
Let's look at a simpler example which, when solved visually, will make it obvious how to sum the series above. Let's add the following series visually:

$$1 + 2 + 3 + ... + 10.$$

If we use "•" to represent a unit, then $1 = \bullet$, $2 = \bullet\bullet$, $3 = \bullet\bullet\bullet$, et cetera. Determining the sum from $1$ to $10$ is equivalent to counting the dots in the following pattern:



Just draw the same number of dots, but upside-down, and we obtain the following picture:



It's easy to count the dots in the pattern above: there are $10$ rows of $11$ dots, for a total of $10 \times 11 = 110$. This is twice as many as we want, however, so there are $110/2 = 55$ dots in the original triangular pattern.

2. Find the sum $1 + 2 + \cdots + n$, where $n$ is an arbitrary positive integer.

   Solution: Item 1

3. Find the sum $1 + 3 + 5 + \cdots + 1001$. In other words, sum the odd numbers from 1 to 1001.

   Solution: Item 2

4. Find a general formula for the sum $1 + 3 + 5 + \cdots + (2n + 1)$.

   Solution: Item 3

5. Find the sum $7 + 10 + 13 + \cdots + 307$.

   Solution: Item 4

6. Find the sum of a general arithmetic series:

$$a + (a + d) + (a + 2d) + \cdots + (a + nd).$$

   Solution: Item 5

7. Find the sum of $1 + 2 + 4 + 8 + \cdots + 128$. (Each term is double the previous.)

   Solution: Item 12

8. Find the sum of $1 + 2 + \cdots + 2^n$.

   Solution: Item 12

9. Find the sum of $3 + 6 + 12 + 24 + \cdots + 3 \cdot 2^n$.

   Solution: Item 13

10. Find the sum of $1/2 + 1/4 + 1/8 + \cdots + 1/256$.

    Solution: Item 14

11. Find the infinite sum: $1/2 + 1/4 + 1/8 + 1/16 + \cdots$.

    Solution: Item 15

12. Find the sum of $a + ar + ar^2 + ar^3 + ar^4 + \cdots$.

    Solution: Item 16

13. Find the sum of $a + ar + ar^2 + ar^3 + ar^n$.

    Solution: Item 17

14. (Telescoping series) Find the finite and the infinte sum below (each term has the form $1/(n(n+1))$):

$$1/6 + 1/12 + 1/20 + 1/30 + \cdots 1/2550,$$

$$1/6 + 1/12 + 1/20 + 1/30 + \cdots .$$

    Solution: Item 18

15. (*) Find the sum $1 + 4 + 9 + 16 + \cdots + n^2$. (**) Can you find the sum of the first $n$ cubes? The first $n$ fourth powers?

    Solution: Item 19

16. Can you find a geometric image that helps you visualize the following pattern?

$$\begin{aligned} 1 &= 1^3 \\ 3 + 5 &= 2^3 \\ 7 + 9 + 11 &= 3^3 \\ 13 + 15 + 17 + 19 &= 4^3 \\ 21 + 23 + 25 + 27 + 29 &= 5^3 \end{aligned}$$

    Solution: Item 19

17. Can you use the diagram in Figure 1 to show another way to calculate $1^3 + 2^3 + 3^3 + \cdots + n^3$?

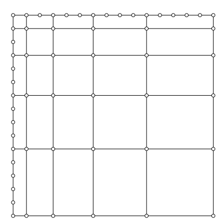


Figure 1: Sum of cubes

    Solution: Item 19

18. Try to draw a diagram with dots that demonstrates that:

$$(n + 1)^3 = n^3 + 3n^2 + 3n + 1.$$

    Hint: one nice solution is three-dimensional. What are the corresponding diagrams in two and one dimension?

    Solution: Item 6

19. (*) Can we extend the idea above to help visualize something about four dimensions?

### 3.2 Equations and their Graphs

Next we'll look at the exact relationship between equations and graphs in two variables, $x$ and $y$. The first couple of exercises seem unrelated, but will help get your mind thinking geometrically about the problems rather than algebraically.

1. Visualization exercises:

   Try to solve these first in your head, without drawing pictures, if possible.

   How many:

   - corners (vertices) does a cube have?
   - faces does a cube have?
   - edges does a cube have?
   - Same questions: how many vertices, faces, edges has a tetrahedron?
   - ... has an octahedron?
   - ... has an Egyptian pyramid?
   - ... has a cube with a corner chopped off?
   - ... have some other shapes?

   Solution: Item 7

2. Describe:

   - intersections of a plane with a sphere
   - intersections of two spheres
   - intersections of a cube with a plane

   Solution: Item 8

3. When we draw a graph of an equation like $y = 3x + 2$, *exactly* what does the graph mean? We often look at graphs of quadratic equations in particular for points where the curve (parabola, for quadratic equations) crosses the $x$-axis. What does this mean?

   Solution: Item 9

4. Examine equations of lines, circles, and parabolas to see find some intuitive reasons why they have the form that they do.

5. What does the graph of this look like:

   $$(x^2 + y^2 - 25)(3x - 2y + 3)(4x^2 - 3x + 2 - y) = 0$$

   Solution: Item 10

6. Visualizing inequalities. For example, how are the graphs of $y = 3x+4$, $y < 3x+4$ and $y > 3x+4$ related? How about $x^2 + y^2 < 25$?

   Solution: Item 11

7. Suppose we are looking for solutions to sets of simultaneous equations in two variables. The number of possible solutions can be imagined by manipulating the graphs in your mind. What sorts of situations can occur with equations having the following sets of graphs?

   - two lines
   - three lines

- line and circle
- two circles
- line and ellipse
- two ellipses
- parabola and circle
- parabola and line
- cubic curve and a line
- cubic curve and a circle
- two cubic curves

8. Making up equations of curves with given properties (like for an exam).

   - Parabolas opening up or down. Left or right.
   - Parabolas symmetric about the y-axis.
   - A cubic polynomial that has roots 1, 2 and 3.
   - A cubic polynomial that has one root, two roots.
   - A line with slope 2/3 that is tangent to the unit circle.

9. Visualizing areas of geometric objects, from "first principles" – in other words, if all we know is that the area of a rectangle with sides of lengths $a$ and $b$ is $ab$, how can we derive the formulas for areas of objects like:

   - a right triangle?
   - any triangle?
   - a trapezoid?
   - a circle?

10. (**) Same question as above, but in three dimensions: If all we know is that the volume of a rectangular solid is $abc$, where the lengths of the sides are $a$, $b$ and $c$, how can we derive formulas for volumes of objects like:

    - a pyramid?
    - a cone?
    - a sphere?
    - a prism?

# 4 Introduction to Symmetry

The most general definition of symmetry might be something like this: "A *thing* is symmetric if when you *do something to it* then it remains *the same*." We usually think of symmetry as a geometric concept, and very often it is, but there are many more aspects to the concept, and we will examine some of them here.

Often the term "*the same*" means that the measurements are the same. If our "*thing*" is a square and we rotate it by $90^\circ$ about its center, then the sides of the rotated square will have the same lengths as in the original square, so we say that a square has a rotational symmetry. There are obviously other symmetries for a square: rotating it by $180^\circ$, $270^\circ$, or flipping it over a diagonal or a line parallel to a side that passes through its center.

There are thus $7$ symmetries of a square, and in fact, we usually say that there are $8$, since we consider the operation of not doing anything to it to be an additional symmetry. This may seem like a strange idea since then everything has as least one symmetry, the "do nothing" operation, but it will make very good sense later on.

You've probably seen wallpaper that has a repeating pattern in the sense that if you slide it in a certain direction, the same drawings would fall on top of copies of themselves, so it would look the same. This, of course, sometimes doesn't work with real wallpaper, since sliding a finite piece of paper over a copy of itself will leave some of it outside the copy, but if the wallpaper were infinite, such a sliding motion would leave everything covered.

# 5 Symmetry Problems

Solutions can be found in Section 8.

Many of the problems below can be solved using standard, brute-force techniques, but every one of them can also be solved in a simpler way using a generalized notion of symmetry.

1. Given a standard $3 \times 3$ tic-tac-toe board, how many essentially different ways are there to make the first move?

2. Can you arrange the numbers $1$ through $9$ on a $3 \times 3$ grid so that every row, column and diagonal adds to the same number? How many ways are there to do this? (This is called a magic square.)

3. A game is played between two players. Begin with the numbers $1$ through $9$ written on the board. Players take turns selecting numbers from the board, and each time a number is selected, it is added to the player's pile and erased from the board. If a player can obtain a set of three numbers that add to $15$ he/she wins. What is a good strategy for this game?

4. (*) Suppose that tic-tac-toe were played on a $4 \times 4$ board, and the goal is to get four squares in a row. How many symmetries would such a board have?

5. (**) Given a $4 \times 4 \times 4$ three-dimensional tic-tac-toe board, how many essentially different ways are there to make the first move? (Hint: there is an "obvious" answer, but the real answer is surprising and amazing.)

6. (**) Given a pentagram with $10$ holes as in Figure 2, fill in the holes with the following $10$ numbers: $1, 2, 3, 4, 5, 6, 8, 9, 10$ and $12$ such that the sum of the numbers on each of the ten straight lines is the same. (Suggested by Harold Reiter.) The "**" rating is for a complete solution; finding one arrangement of numbers that works is not terribly hard, but showing that it is unique (in a sense) is much more difficult.

7. Suppose you decide to sell sudoku puzzles to your local newspaper, but you are too lazy to work out any actual puzzles, so your plan is to steal an existing puzzle and modify it so that it is not easily recognized. What operations can be applied to an existing puzzle so that the resulting puzzle

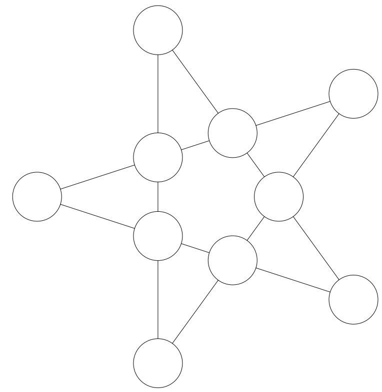


Figure 2: Star Problem

looks different? (Hint: one very easy idea is to place a 2 whereever the original puzzle had a 1 and vice-versa.)

8. If there are 270725 ways to choose four cards from a deck of 52, how many ways are there to choose 48 cards from a deck of 52?

9. A circle is inscribed in an isosceles trapezoid as in Figure 3. (An isosceles trapezoid is a trapezoid where the two non-parallel sides ha ve equal length. In Figure 3, the trapezoid is isosceles if $AD = BC$.) If segment $AB$ has length $l$ and segment $CD$ has length $L$, how long are the other two sides, $BC$ and $DA$?

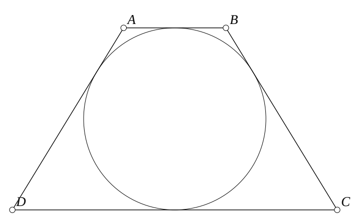



Figure 3: Isosceles Trapezoid

10. Given the following system of two equations and two unknowns, where the numbers $a, b, c, d, e$ and $f$ are constant:

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

Suppose an oracle tells you that for any (well, *almost* any[1]) set of values for $a, b, c, d, e$ and $f$ that the solution for $x$ is given by:

$$x = \frac{ce - bf}{ae - bd}.$$

How can you find the value of $y$, with minimal effort?

11. What is the relationship between the following pairs of graphs: $y = x^2$ and $x = y^2$? How about $x^3 - y^3 - xy = 0$ and $y^3 - x^3 - yx = 0$?

[1]To be precise, for any values such that $ae - bd \neq 0$.

12. What sorts of symmetries can you find in the graphs of the following equations. For example, which ones will be symmetric about the $x$-axis, the $y$-axis, et cetera. What other symmetries can you find?

    - $y = x^2$.
    - $y = x^3$.
    - $y = x^n$, where $n$ is a positive integer.
    - $x^2 + y^2 = 25$.
    - $x^2 + 3y^2 = 25$.
    - $y = 1/x$.
    - $x^2 - y^2 = 1$.

13. What can you say about the graph of a function $f$ that satisfies the following conditions:

    - What if $f(x) = f(-x)$, for all $x$?
    - What if $f(x) = -f(x)$, for all $x$?
    - What if $f(x) = f(x + 2)$, for all $x$?
    - If $f$ is *any* function, what can be said about the graph of the function $f(x^2)$?

14. There are two piles of coins on a table, each of which originally contains 10 coins. A game is played by two people who alternately select a pile and remove some number of coins (at least one) from that pile. The player who removes the last coin from the table wins. Does the first or second player have a winning strategy?

15. Consider the following game. Begin with an empty circular (or rectangular) table. Players alternate moves, and when it is your turn to move, you must place a quarter flat on the table. If there is no space left to do so, you lose. Does the first or second player have a winning strategy?

16. Two players take turns placing bishops on a standard $8 \times 8$ chessboard, but once a bishop is placed, it is not moved, and no bishop can be placed on a square which is attacked by a bishop already placed. The first person who is unable to place a bishop on the board loses. Which player has a winning strategy? (A bishop attacks all the squares that can be reached from its current square by moving along a diagonal in any direction.)

17. A master chess player agrees to play against two novices as follows. The master will play the white pieces on one board and black on the other. He will make his move on the white board, then his opponent on the other board will make his first move with white. From then on, the novices will wait until the master responds, and then one of the novices will make a move. How can the novices assure one win and one loss or two draws?

18. If you flip a fair coin 123 times, at the end are you more likely to have more heads or more tails?

19. An urn contains 500 red balls and 400 blue balls. Without looking at them, 257 balls are removed from the urn and discarded. Finally, a single ball is drawn from the urn. What is the probability that it is red?

20. Find the area under the curve $\cos^2 x$ from $x = 0$ to $x = \pi/2$.

21. You have a cup of coffee and an identical cup of cream. Both contain the same amount of liquid. You take a tablespoon of cream and put it in the coffee. It is then mixed thoroughly and a tablespoon of the resulting mixture is added back to the cream. Is there now more cream in the coffee or more coffee in the cream? What if you don't mix thoroughly before you return the tablespoon of mixture to the coffee cup?

22. A farmer with a bucket needs to water his horse. Both are on the same side of a canal that runs in a straight line. The farmer and his horse are on the same side of the canal, but the farmer needs to go to the river first to fill the bucket before he takes it to his horse. At what point on the canal should he collect the water to minimize the total distance he travels?

23. Suppose your cue ball on a normal rectangular billiard table is at point $P$ and the target ball is at point $Q$. Is it possible to hit the target after bouncing off one cushion? Two cushions? Three? How can you figure out which direction to hit the cue ball to achieve these results. (Assume that the cue ball does a "perfect" bounce each time, with the angle of incidence equal to the angle of reflection. Also assume that the table dimensions are exactly $2:1$.)

24. In a room with rectangular walls, floor and ceiling, if a spider is on one of the surfaces and the fly on another, what is the shortest path the spider can take to arrive at the fly, if the fly does not move? (The answer, of course, will depend on the dimensions of the room, and upon where the spider and the fly initially start. What we're searching for is a method to find the solution for any room size and any initial positions of the spider and the fly.)

25. (*) If you build an elliptical pool table and you strike a ball so that it passes through one of the ellipse's foci, then after it bounces off a cushion, it will pass through the other focus. Show that this is true, based (loosely) on what you learned from the farmer and his horse a couple of problems ago. Remember that an ellipse is defined to be the set of all points such that the sum of their distances to the two foci is constant. Hint: what would the shape of a river be so that it doesn't matter where the farmer goes to get his water?

26. (*) Fagnano's problem. Show that in any acute-angled triangle, the triangle of smallest perimeter that can be inscribed in it is the so-called "pedal triangle" whose vertices are at the feet of the altitudes of the given triangle. In Figure 4, $\triangle DEF$ is the pedal triangle for $\triangle ABC$. What happens if the given triangle contains a right angle or an obtuse angle?

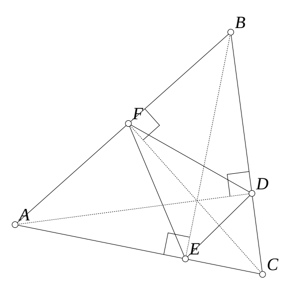



Figure 4: Fagnano's Problem

27. Fifteen pennies are placed in a triangular shape as shown in Figure 5. Many sets of three centers of those pennies form the vertices of equilateral triangles, two samples of which are illustrated in the figure. Is it possible to arrange the pennies in such a manner that no set of penny centers that form an equilateral triangle are all heads or all tails?

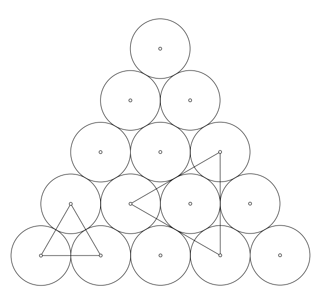


Figure 5: Fifteen Pennies

28. Add the whole numbers from 1 to 100.

29. Find the value of $x > 0$ which minimizes the function $f(x) = x + 1/x$.

30. What is the area of the largest rectangle that can be inscribed in a circle of radius 1?

31. In a triangle with sides 1, 1 and $x$, find the value of $x$ that maximizes the area.

32. Give some strong evidence that an equilateral triangle is the triangle of largest area that can be inscribed in a circle. What is the largest quadrilateral that can be so inscribed? The largest $n$-sided figure?

33. If you have a million distinct points inside a circle, can you find a line that divides them such that there are exactly half on each side?

34. Find all whole number values $a$, $b$ and $c$ such that $a + b + c = abc$.

35. A cube is built with wire edges as in Figure 6. If wires are connected to opposite corners of the cube and a one-ampere current is passed through, how much current flows through each of the edges. (Not every edge will have the same current passing through it.)

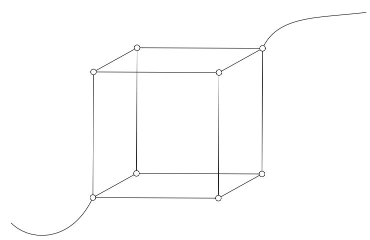


Figure 6: Wire cube

36. (*) A square metal plate has three sides held at a temperature of 100 degrees and the fourth at zero degrees. What's the temperature at the point in the center of the plate[2]?

37. (*) An infinite square mesh of wire (a small part of which is shown in Figure 7) extends in every direction. All grid lengths are equal, and all the wire has the same resistance per unit length. Two wires are connected to adjacent grid points $A$ and $B$ and a one-ampere current enters through $A$ and leaves through $B$. What is the current through $AB$? Note that the electrons will follow many paths, with more following the shorter paths since the resistance is smaller.

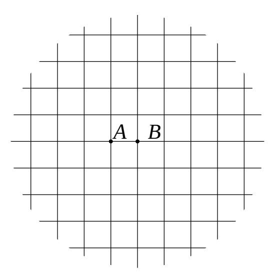



Figure 7: Inifinte wire mesh

[2]This and the following problem depend on a little bit of physics. Solutions to the heat equation (and to electrical circuits in the next problem) often satisfy the condition of superposition, meaning that a set of solutions can be added together to make the final solution. Both of these problems are of that sort.

38. Evaluate the following three expressions using a (translation) symmetry observation:

$$\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$$

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

39. Solve for $x$:

$$2 = x^{x^{x^{\cdots}}}.$$

(*) Solve for $x$:

$$4 = x^{x^{x^{\cdots}}}.$$

What is going on here?

40. How quickly can you expand the following product?

$$(x+y)(y+z)(z+x)?$$

What is different about the product?

$$(x-y)(y-x)(z-x)?$$

41. If $\{x=1, y=2, z=3\}$ is a solution for the following set of equations, find five more solutions. (*) Find all whole-number solutions for $x$, $y$ and $z$ in the equations below:

$$\begin{aligned} x+y+z &= 6 \\ x^2+y^2+z^2 &= 14 \\ xyz &= 6 \end{aligned}$$

42. (**) Find all whole-number solutions for $w$, $x$, $y$ and $z$ in the following system of equations:

$$\begin{aligned} w+x+y+z &= 10 \\ w^2+x^2+y^2+z^2 &= 30 \\ w^3+x^3+y^3+z^3 &= 100 \\ wxyz &= 24 \end{aligned}$$

# 6 A Tiny Introduction to Group Theory

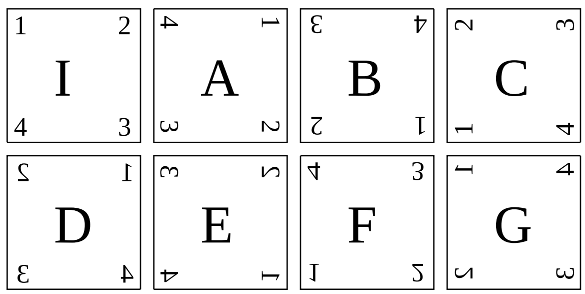
The basic idea of symmetry is that there are certain operations you can apply to a situation, and (in some sense) the situation remains the same. This idea can be formalized via a mathematical construct called "Group Theory."

As a concrete example, consider the symmetries of a perfect square. If you rotate the square about its center by 90°, 180°, or 270°, the result will look exactly the same. Similarly, you could turn the square over, or you could turn it over and apply any of the above rotations and the resulting square would look exactly the same. Thus there are 7 movements you can apply to the square with the result looking identical.

It turns out to be very useful to consider one additional symmetry "operation"; namely, don't do anything to it. That also leaves it looking the same, right? So there are in fact 8 symmetry operations that can be applied to the square.

A nice thing about these operations is that if we do one of them and then follow that with another applied to the transformed square we obtain another symmetry operation. (That is one reason why it is useful to consider the "do nothing," or "identity" operation to be a symmetry: if the rotate by 90° and then by 270° we effectively return it to its original configuration, so the net effect is that we have done nothing. By including the "do nothing" (which we will call the "identity" operation from now on), the set of symmetry operations is closed in the sense that if you apply one and then another the result is a third.

If we number the four vertices of the square simply so we can keep track of where the corners move after a symmetry operation, then from the starting position (shown in the upper left corner) we can transform it to any of the 8 positions below. (One of the transformations, of course is the identity, meaning the resulting position is the same as the starting position:




The large letters in the middle of each square name the transformation. "$I$" stands for the identity, "$A$" for rotation by 90° clockwise, et cetera.

Now we can form a sort of multiplication table for the transformations. For example, if we repeat $A$ twice, it's the same as doing a single $B$ operation (two 90° clockwise transformations is the same as a single 180° transformation).

One thing to note is that the four top transformations are all rotations and the bottom four are all reflections. For example, $D$ is the reflection across a vertical line passing through the center, $E$ is a reflection across the line connecting points 2 and 4, et cetera.

We will indicate with the symbol $*$ the operation of combining two transformations, so for example, $A * A = B$, $A * C = I$, and $A * F = E$. Note that $A * F$ means " first perform $A$, then perform $F$, not the other way around. In fact, $F * A = G \neq A * F$. In other words, the operation called $*$ is not commutative! In fact, here is a multiplication table for $*$. To find the value of $A * F$, look in row $A$ and column $F$. To find $F * A$ you would do the opposite: look in row $F$ and column $A$.

| $*$ | $I$ | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ |
|---|---|---|---|---|---|---|---|---|
| $I$ | $I$ | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ |
| $A$ | $A$ | $B$ | $C$ | $I$ | $G$ | $D$ | $E$ | $F$ |
| $B$ | $B$ | $C$ | $I$ | $A$ | $F$ | $G$ | $D$ | $E$ |
| $C$ | $C$ | $I$ | $A$ | $B$ | $E$ | $F$ | $G$ | $D$ |
| $D$ | $D$ | $E$ | $F$ | $G$ | $I$ | $A$ | $B$ | $C$ |
| $E$ | $E$ | $F$ | $G$ | $D$ | $C$ | $I$ | $A$ | $B$ |
| $F$ | $F$ | $G$ | $D$ | $E$ | $B$ | $C$ | $I$ | $A$ |
| $G$ | $G$ | $D$ | $E$ | $F$ | $A$ | $B$ | $C$ | $I$ |

It is a good idea to check a few of the values in the table so that you can understand how the $*$ operation works.

For a set of transformations and their "product" to behave in a reasonable way, we need to have the following conditions be true:

1. The set is closed in the sense that the product of any two of them yields another member of the set.
2. For any transformation, there is the opposite transformation (the inverse) that undoes the transformaton.
3. There is an identity transformation that does nothing. In other words, when combined with any other transformation, the result is simply the other transformation.
4. Associativity: if $u$, $v$, and $w$ are any three transformations, then $u * (v * w) = (u * v) * w$.

Note that the rules above do not require that the operation $*$ be commutative. In other words, it may be possible that $a * b \neq b * a$.

In fact, the four conditions above are basically exactly the four conditions that a mathematician uses to define a "group". Here's what the mathematician might say:

If $S$ is a set of elements, and $*$ is a binary operation mapping $S \times S$ into $S$, then $(S, *)$ is called a group if the following conditions apply:

1. There is and element $e \in S$ such that for every element $a \in S$, $a * e = e * a = a$. The element $e$ is called the "identity" of the group.
2. For any element $s \in S$ there exists an element $s^{-1} \in S$ such that $s * s^{-1} = s^{-1} * s = e$. The element $s^{-1}$ is called the inverse of the element $s$.
3. If $a$, $b$, and $c$ are any three elements of $S$, then $a * (b * c) = (a * b) * c$.

Notice that the first condition in the previous description is automatically true since we said that $*$ maps pairs of elements of $S$ back into $S$.

If we consider sets of symmetries of almost anything, that set of symmetries forms a group. Groups can be finite or infinite, they may be commutative or not, and you can spend the rest of your life studying them. Here are a few examples of mathematical objects that are groups:

1. The symmetries of any geometric object (polygons, polyhedra, graphs, et cetera). Note that if an object has no symmetries, then its group of symmetries consists of a single element, the identity. (Check that such a trivial group satisfies all the conditions above. The set $S = \{e\}$ and the binary operation says that $e * e = e$.)
2. The integers from 0 to $n - 1$ under addition modulo $n$. This is a commutative group.
3. The integers from 1 to $p - 1$ where $p$ is a prime number under multiplication modulo $p$. This is a commutative group.
4. The integers or real numbers under addition. The identity is 0 and the inverse of any number $n$ is $-n$. This is an abelian (commutative) infinite group. Why is the set of integers under multiplication *not* a group?
5. The real numbers or the rational numbers where 0 is omitted under the operation of multiplication. These are two other infinite groups that are commutative. The identity is 1 and the inverse of a number $x$ is $1/x$.
6. The set of $n \times n$ matrices with non-zero determinant form a group under matrix multiplication. This is infinite and non-commutative. Note that if a square matrix has a non-zero determinant, it has a multiplicative inverse. The identity, of course, is the identity matrix which has 1 in every diagonal position and zeroes elsewhere.

One very interesting thing about groups is that many of them contain subgroups: collections of elements from the original group which form a group themselves under the same group operation.

For example, let's consider again the group of symmetries of a square that we enumerated above. Suppose that the square was painted different colors on both sides, and that in addition to having the transformed square fit exactly where the original one did, the colors also had to match. In other words, you're not allowed to turn the square over. In this case, only the rotations (the operations labeled $I$, $A$, $B$, and $C$) are valid symmetries, and the corresponding multiplication table for that subgroup looks like this (which is just the upper right corner of the previous multiplication table):

| $*$ | $I$ | $A$ | $B$ | $C$ |
|---|---|---|---|---|
| $I$ | $I$ | $A$ | $B$ | $C$ |
| $A$ | $A$ | $B$ | $C$ | $I$ |
| $B$ | $B$ | $C$ | $I$ | $A$ |
| $C$ | $C$ | $I$ | $A$ | $B$ |

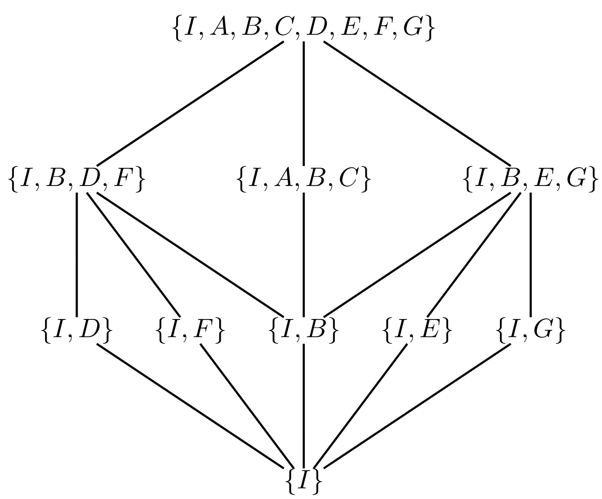
This subgroup happens to be commutative.

Are there any other subgroups? It turns out that there are. In fact, since we don't require that the subset of elements have fewer elements than the original group, the original group is technically a subgroup of itself. Every group has a trivial subgroup consisting of only the identity, so $\{I\}$ is a valid subgroup.

Notice that the following are also subgroups: $\{I, D\}$, $\{I, E\}$, $\{I, F\}$, and $\{I, G\}$. The elements $D, E, F$, and $G$ simply flip the square over about a particular axis, so flipping again just brings the square back to its original position. We can also check that $\{I, B\}$ is a subgroup: rotate the square half-way around, and if you do that twice, you're back to where you started.

There are, in fact, two more subgroups: $\{I, B, D, F\}$ and $\{I, B, E, G\}$. (To check that these are subgroups basically all you have to do is make sure that inverses exist for each element and that the subset is closed under the operation $*$. Associativity is inherited from the larger group, and if an element and its inverse are part of the set and it is closed, then their product, the identity, is also in the set.)

These are all the subgroups: the group itself, the trivial subgroup, five subgroups with two elements, and three subgroups with four elements. In fact, the subgroups of a group always form a lattice, as illustrated below. A line from a lower group to one above it indicates that the group below is a subgroup of the group above:




It is interesting to note that the size of every subgroup happens to divide the size of the original group. (The original group in this case has $8$ elements and the subgroups have sizes $1, 2, 4$, and $8$.) This turns out to be true for any group, and we will end our brief introduction to group theory by proving that fact, at least assuming that the larger group has a finite number of elements, and therefore, so does its subgroup.

Suppose that $G$ is a group and $H$ is a subgroup of $G$. If $g \in G$ is any element of $G$ then we define $gH$ to be:

$$gH = \{g * h | h \in H\}.$$

The set $gH$ is called a “left coset” of $H$.

First we show that if $g_1 \in G$ and $g_2 \in G$ then either the cosets $g_1H$ and $g_2H$ are identical or they have no elements in common. If they have no elements in common, then we are done, so suppose there is at least one element in common. That means there exist $h_1 \in H$ and $h_2 \in H$ such that $g_1h_1 = g_2h_2$.

Now let $g_1h_3$ be any element in $g_1H$. We will show that it is also in $g_2H$, so the two cosets have to be identical.

Since $h_1 \in H$ and $H$ is a subgroup (and hence a group itself), $h_1^{-1}$ is also in $H$, so we can multiply both sides of the equation $g_1h_1 = g_2h_2$ by $h_1^{-1}$ on the right, yielding:

$$g_1h_1h_1^{-1} = g_2h_2h_1^{-1}$$

or

$$g_1 = g_2h_2h_1^{-1}.$$

Since $h_3 \in H$ and $H$ is a group, we know that $h_2h_1^{-1}h_3 \in H$ and therefore:

$$g_2h_2h_1^{-1}h_3 \in g_2H,$$

but

$$g_2h_2h_1^{-1}h_3 = g_1H_3 \in g_1H$$

so every element of $g_1H$ is a member of $g_2H$. Exactly the same argument can be used in the other direction (in fact, this is a sort of an argument by symmetry: swap the 1’s and the 2’s in the subscripts, and obtain another proof).

So what we know so far is that any two cosets of $H$ are either identical or have no elements in common.

If the cosets are all the same size, then since $eH = H$, they all have the same number of elements as $H$, and therefore the size of group $G$ has to be a multiple of the size of subgroup $H$.

So what we need to prove is that if $g \in G$ is any element of $G$, then if $h_1 \in H$ and $h_2 \in H$ are two different members of $H$ then $gh_1 \neq gh_2$. If $gh_1 = gh_2$ multiply both sides by $g^{-1}$ on the left and obtain $g^{-1}gh_1 = g^{-1}gh_2$, or $h_1 = h_2$. Thus all cosets of $H$ have the same size; namely, the size of $H$, and since all the cosets together form the group $G$, we know that the size of $G$ is a multiple of the size of $H$.

# 7 Solutions to Visualization Problems

1. Find the sum $1 + 2 + \cdots + n$, where $n$ is an arbitrary positive integer.

   Two triangular sets from 1 to $n$ are constructed which, when put together, form a rectangle that's $n \times (n+1)$. Thus twice the total is $n(n+1)$, so:

   $$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

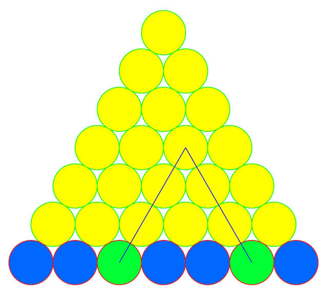
   If you know a little bit about counting and binomial coefficients, here is another really nice way of visualizing the situation. In combinatorics, it turns out that the formula for $\binom{n}{2}$, which is the number of ways to choose 2 objects from a set of $n$ objects, is:

   $$\binom{n}{2} = \frac{n(n-1)}{2}.$$

   Equivalently, the number of ways to choose 2 objects from a set of $n+1$ objects will be given by:

   $$\binom{n+1}{2} = \frac{n(n+1)}{2},$$

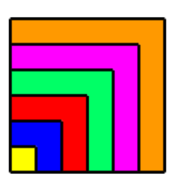
   Which is what we want to prove is the value of $1 + 2 + 3 + \cdots + n$.

   In the figure above, the yellow circles represent $1 + 2 + 3 + \cdots + n$, where $n = 6$ in this case. So the blue and green row contains $n+1$ circles. For any choice of two circles in that bottom row (for example the two green ones), we can draw lines parallel to the triangle's sides that will intersect in a unique yellow circle. Every yellow circle determines exactly one possible choice of a pair and vice-versa, so the number of yellow circles must be equal to the number of ways to choose exactly two circles from the bottom row.

2. Find the sum $1 + 3 + 5 + \cdots + 1001$. In other words, sum the odd numbers from 1 to 1001.

   In the illustration below, starting from the yellow square in the lower-left corner, we see successive "V"-shaped groups of squares. Imagine that the entire square below is divided into small squares that are the same size as that yellow square. Notice that each group as we go out from the yellow square has two more small squares in it than the previous, so the number of small squares in the groups are: $1, 3, 5, 7, \ldots$. When you group all of the "V"-shaped regions together, you obtain a square.

If the outer-most "V" has $n$ little squares in it, what is the size of the large square? Note that $n$ is odd, so has the form $n = 2k + 1$. If you think of the "1" square as the one at the tip of the "V", then there must be $k$ horizontal and $k$ vertical squares in the "V" for a total of $2k + 1$. From that, it is easy to see that the large square has a side of $k + 1$, or an area of $(k + 1)^2$, which is the total number of small squares contained in it. In our original problem, we wanted to sum:

$$1 + 3 + 5 + \cdots + 1001,$$

and $1001 = 2 \times 500 + 1$, so if we had drawn a giant square corresponding to this sum, its side would be 501 and its area: $(501)^2 = 251001$. Thus:

$$1 + 3 + 5 + \cdots + 1001 = (501)^2 = 251001.$$

This problem can also be solved in exactly the same way as we summed the first two examples in this section, assuming that you are careful about counting the number of dots in the length and width of the resulting rectangle.

3. Find a general formula for the sum $1 + 3 + 5 + \cdots + (2n + 1)$.

   Using exactly the same argument as above, we'll have a square with a side of $(2n + 1 + 1)/2 = (2n + 2)/2 = n + 1$, so the sum is:

$$1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2.$$

4. Find the sum $7 + 10 + 13 + \cdots + 307$.

   Again, all we need to do is reverse the list and add. There will be a number of terms with sum $7 + 307 = 314$. The only problem is to count how many there are. Since we add 3 each time, it's like adding $0 + 7$ up to $300 + 7$, so how many multiples of 3 are there between 0 and 300. You have to be a little careful: the answer is 101. Often people just divide 300 by 3 and obtain 100. It's true that there are 100 steps between 0 and 300, so there is a number after each step, for 100 of them, but there's also the 0 before the first step. That makes 101 total. Thus:

$$7 + 10 + 13 + \cdots + 307 = (314 \times 101)/2 = 31714/2 = 15857.$$

5. Find the sum of a general arithmetic series:

$$a + (a + d) + (a + 2d) + \cdots + (a + nd).$$

   As with the case above, after you reverse the series and add, each term will have the sum $(2a + nd)$. There are $(n + 1)$ terms, so the sum of a general arithmetic series is:

$$a + (a + d) + (a + 2d) + \cdots + (a + nd) = \frac{(2a + nd)(n + 1)}{2}.$$

   It's often a good idea to check these general formulas with a specific case or two to make sure there wasn't a foolish error, so let's check the sum in the previous problem. Obviously, $a = 7$, $n = 100$, and $d = 3$, so the sum should be:

$$\frac{(14 + 300)(101)}{2} = 15857.$$

6. Try to draw a diagram with dots that demonstrates that:

$$(n + 1)^3 = n^3 + 3n^2 + 3n + 1.$$

   Hint: one nice solution is three-dimensional. What are the corresponding diagrams in two and one dimension?

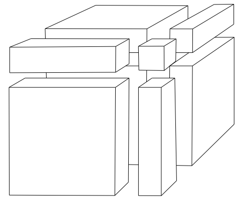


Figure 8: $(n + 1)^3 = n^3 + 3n^2 + 3n + 1$

See Figure 8. Imagine that a large cube with all three sides equal in length to $n+1$ is sliced as in the figure, where the thin slices have width 1. The original cube has volume $(n + 1)^3$, and it is cut into eight pieces. One is a cube of side $n$ with volume $n^3$, three are plates having volume $n \cdot n \cdot 1 = n^2$ (since the thickness is 1), three are rods with volume $n \cdot 1 \cdot 1 = n$, and there is a single small cube of volume $1^3 = 1$. Adding the eight volumes together shows us that the total volume is:

$$(n + 1)^3 = n^3 + 3n^2 + 3n + 1.$$

7. How many

   - corners (vertices) does a cube have? Answer: 8
   - faces does a cube have? Answer: 6
   - edges does a cube have? Answer: 12
   - Same questions: how many vertices ($V$), faces ($F$), edges ($E$) has a tetrahedron?
     Answer: $V = 4, F = 4, E = 6$.
   - ... has an octahedron?
     Answer: $V = 6, F = 8, E = 12$.
   - ... has an Egyptian pyramid?
     Answer (counting the base): $V = 5, F = 5, E = 8$.
   - ... has a cube with a corner chopped off?
     Answer: $V = 10, F = 7, E = 15$.
   - ... have some other shapes?
     Obviously, the answers depend on the solid, but assuming the solid has no holes, the answer has to satisfy: $V - E + F = 2$. This is known as Euler's Theorem.

8. Describe:

   - intersections of a plane with a sphere

     If the plane touches the sphere at a point, then there is a one-point intersection. Imagine touching a flat plate to a ball.

     If the plane cuts the sphere in more than one point, it must cut a circle. The largest circle it can cut will occur when the plane passes through the center of the sphere. If this occurs, the circle of intersection is called a "great circle". To help visualize this, think of the sphere as the earth, and look at the lines of constant latitude (constant distance north or south of the equator). All are circles and if you look at a globe of the earth, and look straight down on the north pole, all the constant-latitude lines are circles. Imagine a plane perpendicular to the axis of the north-south pole passing through the earth. At every stage it will cut a circle of constant latitude.

     A few great circles on the earth are the equator or the lines of constant longitude, but there are many others. If you are forced to stay on the surface of the earth (in a boat, for example) the shortest path is along a great circle. To find that great circle, consider the origin and destination as two points and find the plane passing through those two points and the center of the earth. The path will be along the great circle which is the intersection of that plane and the surface of the earth.

- intersections of two spheres

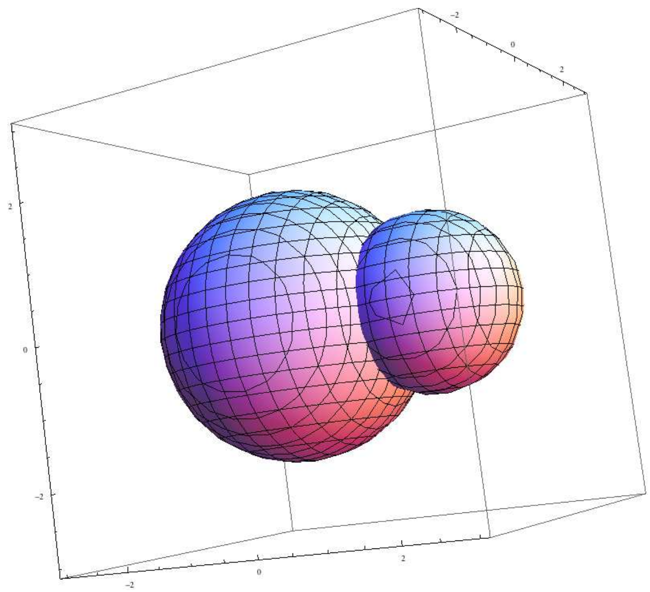


Figure 9: Two intersecting spheres

See Figure 9. As with the plane and the sphere, two spheres can touch at a point. If the intersection is larger than a point, it will be a circle (or, if the two spheres are identical, it will be the entire sphere). To see why it is a circle, imagine a line connecting the centers of the spheres, and slide a plane along that line, perpendicular to it, until it passes through a common point of the two spheres. The line will cut a circle on both spheres of identical size, so it must the the same circle on both of them.

- intersections of a cube with a plane

This is actually a *very* difficult visualization exercise. If the plane just touches a vertex, you can get just that point. If the plane intersects an edge, you can get a line segment the length of the edge. If the plane cuts into the cube, you can get all sorts of odd-shaped polygons from triangles to hexagons. In fact, it's possible to get a perfect hexagon.

If the plane just cuts the tip of a vertex, you can get a triangle. A cut near the vertex perpendicular to the axis connecting that vertex to the opposite one on the cube will yield an equilateral triangle. If the plane is tilted, many more triangles can be obtained, ranging from equilateral to very long, skinny ones.

If you cut parallel to a face, you'll obtain a perfect square. If you imagine hanging the cube by a vertex and cutting half-way between the top and bottom vertices perpendicular to that vertex-vertex axis, that will make a perfect hexagon. By tilting away from that axis, you'll get pentagons.

9. When we draw a graph of an equation like $y = 3x + 2$, *exactly* what does the graph mean? We often look at graphs of quadratic equations in particular for points where the curve (parabola, for quadratic equations) crosses the $x$-axis. What does this mean?

Every point on the graph represents a different solution to the equation. Take any point of the graph and it will have an $x$ and a $y$ coordinate, each of which is just a number. If you let $x$ be the $x$-number and $y$ the $y$-number and plug those into the equation, the equation will be satisfied.

When a graph crosses the $x$-axis that means that the $y$ coordinate is zero. Often you are trying to solve an equation like $0 = x^2 - 7x + 2$ which is the same thing as trying to find a value of $x$ that makes the parabola $y = x^2 - 7x + 2$ pass through the $x$-axis. When $y = 0$, that's the $x$ you want. (Or, if more than one $x$ value makes $y = 0$, it simply means the curve passes through the $x$-axis more than once, so the equation has multiple solutions.)

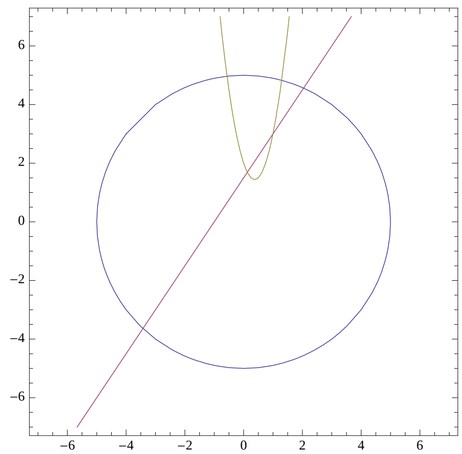



Figure 10: Graph of $(x^2 + y^2 - 25)(3x - 2y + 3)(4x^2 - 3x + 2 - y) = 0$

10. See Figure 10. We are trying to find the places where the product of three expressions is zero. If a product of numbers is zero, then one or more of those numbers must be zero. So we will have the product equal to zero if any of the following three equations is true:

$$x^2 + y^2 - 25 = 0$$
$$3x - 2y + 3 = 0$$
$$4x^2 - 3x + 2 - y = 0$$

The first is just the equation of a circle of radius 5 centered at the origin; the second is the equation of a straight line, and the third is the equation of a parabola. The figure shows all the solutions for all three.

Another way to visualize this is to imagine the surface of a three-dimensional plot of:

$$z = (x^2 + y^2 - 25)(3x - 2y + 3)(4x^2 - 3x + 2 - y).$$

This will be a complicated three-dimensional plot, but what we are interested in is the intersection of this plot with the plane $z == 0$. This is illustrated in Figure 11.

11. How are the graphs of $y = 3x + 4$, $y < 3x + 4$ and $y > 3x + 4$ related? How about $x^2 + y^2 < 25$?

The line $y = 3x + 4$ represents all of the $(x, y)$ coordinates where the equation is exactly satisfied. Almost always, on one side of that line, the value of $y$ will be smaller than $3x + 4$ and on the other side, it will be greater.

If we want the graph of $y < 3x + 4$ to represent all the points where this inequality is true, that graph will consist of all the points on one side of the line (but not including the line) $y = 3x + 4$. Similarly, the graph of $y > 3x + 4$ will consist of all the points on the other side of the line. All you need to do to find out which side of the line is included is to test a single point on one side of the line, say the point where $x = y = 0$. Since $0 < 4$ we know that the side of the line that includes the origin is the side that represents the graph of $y < 3x + 4$.

For the inequality $x^2 + y^2 < 25$, it's easiest to look first where there is equality; namely: $x^2 + y^2 = 25$. This is a circle of radius 5 centered at the origin, $(0, 0)$. By checked the state of the equation with a point inside and outside the circle, we see that if $x = y = 0$ that $x^2 + y^2 < 25$, so the original inequality is satisfied for all the points inside (but not on) the circle.

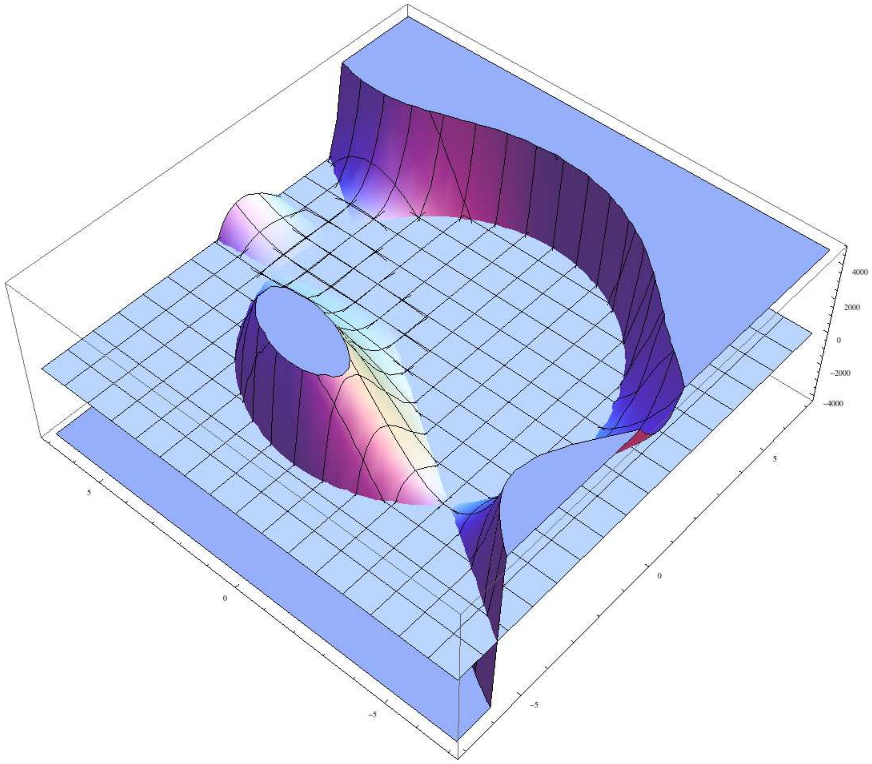



Figure 11: Graph of $(x^2 + y^2 - 25)(3x - 2y + 3)(4x^2 - 3x + 2 - y) = 0$

12. See Figure 12. Begin with the small square of area $1$ near the upper left corner of the figure. Then rectangles and squares having areas $2, 4, 8$, and so on, are added alternately below and to the right of the original square. Thus, for example, the sum $1 + 2 + 4 + 8$ is represented by a square in the upper right that is missing a single tiny square at its upper left corner.

    Assuming the little missing square were actually there, each rectangle or square added doubles the previous area, so if there are $n$ terms in: $1 + 2 + 4 + \cdots + 2^{n-1}$, the sum of those terms will be $2^n - 1$. The "$-1$" subtracts off the area of the upper left corner that is missing.

    From these observations, we can see that:

    $$1 + 2 + 4 + 8 + \cdots + 128 = 256 - 1 = 255,$$

    and

    $$1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1.$$

13. This is based on the solution in Item 12. Note that every term in this series is exactly 3 times as big as the term in the series:

    $$1 + 2 + 4 + 8 + \cdots ,$$

    so the sum will be 3 times as large:

    $$3 + 6 + 12 + 24 + \cdots + 3 \cdot 2^n = 3(2^{n+1} - 1).$$

14. See Figure 13. In the figure, imagine that the area of the entire square is 1. It is successively divided, first into halves, then one of the halves is divided in half making two quarters, one of the quarters

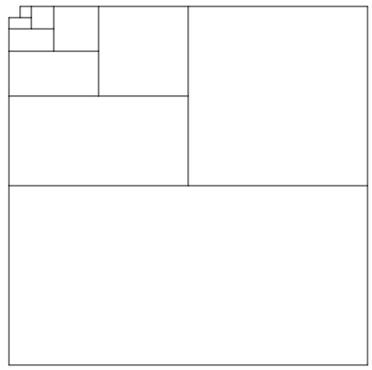


Figure 12: Adding $1 + 2 + 4 + 8 + \cdots$

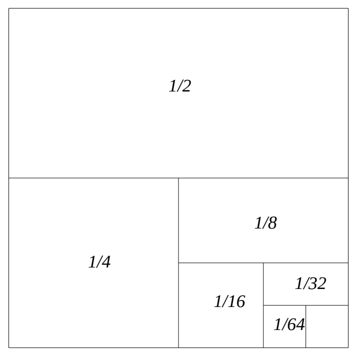



Figure 13: Adding $1/2 + 1/4 + 1/8 + \cdots$

is split in half making two eighths, et cetera. At each stage of the calculation:

$$\begin{array}{l} 1/2 \\ 1/2 + 1/4 \\ 1/2 + 1/4 + 1/8 \\ 1/2 + 1/4 + 1/8 + 1/16 \\ \ldots \end{array}$$

if you look at a set of areas that corresponds to that, the remainder to complete the entire square is simply a copy of the smallest rectangle/square that you obtained. Thus:

$$1/2 + 1/4 + 1/8 + \cdots + 1/256 = 1 - 1/256 = 255/256.$$

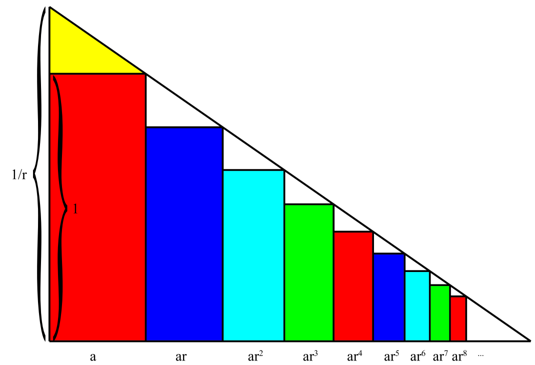
15. See Item 14 and Figure 13. Each additional term in the series:

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \cdots$$

fits inside the large square of area one, but basically cuts the uncovered part in half. Thus, as more and more terms are added, the uncovered part gets as tiny as you want, so the area gets closer and closer to $1$, so it is reasonable to set the infinite sum to $1$. This idea can be made rigorous mathematically (the theory of limits), but this paper is more concerned with visualization, so we will not do so here.

16. Find the sum of $a + ar + ar^2 + ar^3 + ar^4 + \cdots$.

It is actually easier to evaluate the infinite geometric sum than the finite one. See the figure below:




The red, blue, cyan, green, red, et cetera, rectangles have bases equal in length to $a$, $ar$, $ar^2$, $ar^3$, et cetera. The height of the largest (red) rectangle is $1$. The length of the base of the large triangle is $S$, where $S = a + ar + ar^2 + ar^3 + \cdots$. The entire triangle is similar to the small yellow triangle in the upper left. The sides of the yellow triangle are $1/r - 1$ and $a$, while the corresponding sides of the entire triangle are $1/r$ and $S$. Because the two triangles are similar, we have:

$$\frac{a}{1/r - 1} = \frac{S}{1/r},$$

from which it is easy to conclude that:

$$S = \frac{a}{1-r}.$$

The usual way this is proved in textbooks is as follows:

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \cdots \\ rS &= \phantom{a + {}} ar + ar^2 + ar^3 + \cdots \end{aligned}$$

If we subtract the two equations above, we obtain:

$$S - rS = a, \text{ or } S = \frac{a}{1-r}.$$

17. Find the sum of $a + ar + ar^2 + ar^3 + ar^n$.

    An easy way to calculate the finite sum of a geometric series is to begin with the formula for the infinite sum.

$$\begin{aligned} S_1 &= a + ar + ar^2 + \cdots + ar^n + ar^{n+1} + ar^{n+2} + \cdots \\ S_2 &= \phantom{a + ar + ar^2 + \cdots + ar^n + {}} ar^{n+1} + ar^{n+2} + \cdots \end{aligned}$$

    Both $S_1$ and $S_2$ are geometric series, but the first term of $S_2$ is not $a$; it is $ar^{n+1}$. The sum we want is:

$$S = S_1 - S_2 = \frac{a}{1-r} - \frac{ar^{n+1}}{1-r} = \frac{a - ar^{n+1}}{1-r} = \frac{a(1-r^{n+1})}{1-r}.$$

18. (Telescoping series) Find the finite and the infinte sum below (each term has the form $1/(n(n+1))$):

$$1/6 + 1/12 + 1/20 + 1/30 + \cdots 1/2550,$$

$$1/6 + 1/12 + 1/20 + 1/30 + \cdots .$$

The key here is to note that:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

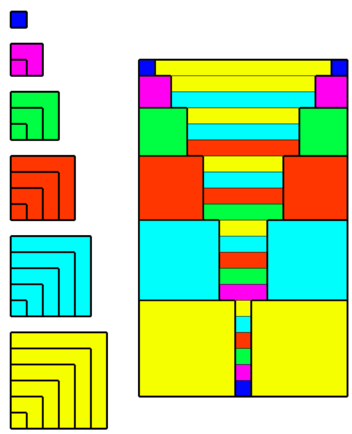
Thus $1/6 = 1/2 - 1/3$, $1/12 = 1/3 - 1/4$, $1/20 = 1/4 - 1/5$ and so on. With this observation, the first sum is equal to this:

$$(1/2 - 1/3) + (1/3 - 1/4) + (1/4 - 1/5) + \cdots + (1/50 - 1/51).$$

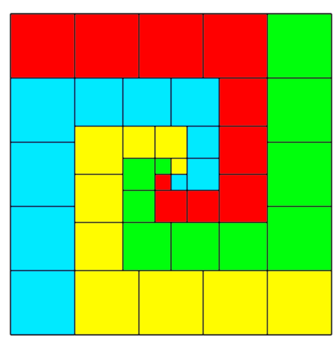
If we regroup, the sum "telescopes", and is equal to $1/2 - 1/51 = 49/102$.

In the case of the infinite sum, the terms cancel all the way down, so the answer is just $1/2$.

19. Here is an image that shows that $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 6(7)(13)/6$. There are three copies of each square, all in the same color. The squares on the left are subdivided as shown and are fit into the larger rectange (which is $13 \times (3 \times 7)$).

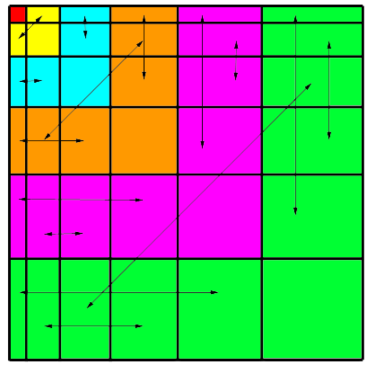


Here is an image that shows that $1^3 + 2^3 + 3^3 + 4^3 = (4 \cdot 5)^2/4$. There are four copies of $1(1^2), 2(2^2), 3(3^2)$ and $4(4^2)$. The large square size is $5 \times 5$. It should be clear that the pattern can be continued:



Here is yet another way to visualize the sum of cubes displayed in the figure below. The side of the large square is $1 + 2 + 3 + \cdots + n = n(n+1)/2$, so its area the square of that. If you put together each pair of rectangles that are connected by a two-headed arrow, each pair forms a square, and you can see that there is one $1 \times 1$ red square, two $2 \times 2$ yellow squares, three $3 \times 3$ cyan squares, et cetera. The total areas sum to $1^3$, $2^3$, $3^3$, et cetera. Thus:

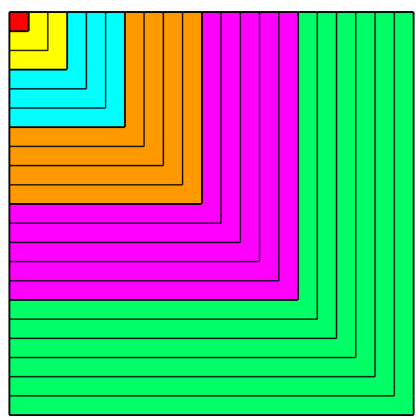
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$



A modification of the drawing above illustrates the fact that:

$$\begin{array}{rcl} 1 &=& 1^3 \\ 3+5 &=& 2^3 \\ 7+9+11 &=& 3^3 \\ 13+15+17+19 &=& 4^3 \\ 21+23+25+27+29 &=& 5^3 \end{array}$$

In the drawing below, each of the colored regions represents cubes: The red region has $1^3$ squares, the yellow region has $2^3$ squares, the cyan region, $3^3$ squares and so on. But each region is also composed of a set of "V"-shaped sets of squares that contain successively-larger sets of odd numbers. The sums of those odd numbers are mathematically represented by the sums above.



We will show how a method used to sum $1+2+\cdots+n$ can be used to sum other sums of powers. Here is a slightly different way to evaluate the simpler sum:

$$\begin{array}{ccccccccc|c} 1 & & & & & & & & & 1 \\ 1 & + & 1 & & & & & & & 2 \\ 1 & + & 1 & + & 1 & & & & & 3 \\ \vdots & & \vdots & & \vdots & & \ddots & & & \vdots \\ 1 & + & 1 & + & 1 & + & \cdots & + & 1 & n \\ \hline n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 & S_1(n) \end{array}$$

We can obtain the same result by summing the row at the bottom or the column on the right. On the right we obtain $1+2+\cdots+n = S_1(n)$. Assuming all the summations are for $i=1$ to $i=n$, rewrite the row on the bottom as:

$$\sum(n-i+1) = \sum n - \sum i + \sum 1 = n^2 - S_1(n) + n.$$

Set the two sums equal:

$$\begin{aligned} n^2 - S_1(n) + n &= S_1(n) \\ n^2 + n &= 2S_1(n) \\ \frac{n^2+n}{2} = S_1(n) \end{aligned}$$

Next we will evaluate a different sum that will allow us to evaluate $S_2(n)$:

$$\begin{array}{ccccccccc|c} 1 & & & & & & & & & S_1(1) \\ 1 & + & 2 & & & & & & & S_1(2) \\ 1 & + & 2 & + & 3 & & & & & S_1(3) \\ \vdots & & \vdots & & \vdots & & \ddots & & & \vdots \\ 1 & + & 2 & + & 3 & + & \cdots & + & n & S_1(n) \\ \hline n\cdot 1 & + & (n-1)\cdot 2 & + & (n-2)\cdot 3 & + & \cdots & + & 1\cdot n & T \end{array}$$

We are not directly looking for the sum $T$ of all the numbers in the table, but can obtain the value of $T$ by summing the row at the bottom or the column on the right. On the right we obtain $S_1(1) + S_1(2) + \cdots + S_1(n) = T$). Again assuming all the summations are for $i = 1$ to $i = n$, rewrite this as:

$$T = \sum S_1(i) = \sum \frac{i(i+1)}{2} = \frac{\sum i^2}{2} + \frac{\sum i}{2} = \frac{S_2(n)}{2} + \frac{n(n+1)}{4}.$$

Similarly, we can rewrite the sum of the bottom row as:

$$T = \sum (n-i+1)i = n\sum i - \sum i^2 + \sum i = n\frac{n(n+1)}{2} - S_2(n) + \frac{n(n+1)}{2}.$$

Set the two sums equal:

$$\begin{aligned} \frac{S_2(n)}{2} + \frac{n(n+1)}{4} &= n\frac{n(n+1)}{2} - S_2(n) + \frac{n(n+1)}{2} \\ 3\frac{S_2(n)}{2} &= \frac{n^2(n+1)}{2} + \frac{n(n+1)}{4} \\ 3\frac{S_2(n)}{2} &= \frac{(2n+1)(n)(n+1)}{4} \\ S_2(n) &= \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Finally, we will evaluate yet another sum that will allow us to evaluate $S_3(n)$. It should be obvious that the same technique can be used, one step at a time, to calculate a formula for $S_4(n)$, $S_5(n)$, $\ldots$.

$$\begin{array}{ccccccccc|c} 1 & & & & & & & & & S_2(1) \\ 1 & + & 4 & & & & & & & S_2(2) \\ 1 & + & 4 & + & 4 & & & & & S_2(3) \\ \vdots & & \vdots & & \vdots & & \ddots & & & \vdots \\ 1 & + & 4 & + & 9 & + & \cdots & + & n^2 & S_2(n) \\ \hline n\cdot 1 & + & (n-1)\cdot 2^2 & + & (n-2)\cdot 3^2 & + & \cdots & + & 1\cdot n^2 & T \end{array}$$

As before, we will evaluate $T$ summing the rows first and then summing the columns first. When the two results are set equal and a little algebra is applied, we obtain the formula for $S_3(n)$. We are not directly looking for the sum $T$ of all the numbers in the table, but can obtain the value of $T$ by summing the row at the bottom or the column on the right. On the right we obtain $S_2(1) + S_2(2) +$

$\cdots + S_2(n) = T$). Again ssuming all the summations are for $i = 1$ to $i = n$, rewrite this as:

$$\begin{aligned} T &= \sum S_2(i) = \sum \frac{i(i+1)(2i+1)}{6} = \frac{\sum i^3}{3} + \frac{\sum i^2}{2} + \frac{\sum i}{6} \\ &= \frac{S_3(n)}{3} + \frac{S_2(n)}{2} + \frac{S_1(n)}{6} \\ &= \frac{S_3(n)}{3} + \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{12}. \end{aligned}$$

Similarly, we can rewrite the sum of the bottom row as:

$$\begin{aligned} T &= \sum (n-i+1)i^2 = n\sum i^2 - \sum i^3 + \sum i^2 \\ &= n\frac{n(n+1)(2n+1)}{6} - S_3(n) + \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Set the two sums equal:

$$\begin{aligned} \frac{S_3(n)}{3} &+ \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{12} \\ &= n\frac{n(n+1)(2n+1)}{6} - S_3(n) + \frac{n(n+1)(2n+1)}{6} \\ 4\frac{S_3(n)}{3} &= \frac{(2n+1)n(n+1)(2n+1) - n(n+1)}{12} - \frac{n(n+1)}{12} \\ 4\frac{S_3(n)}{3} &= \frac{(2n+1)^2 n(n+1) - n(n+1)}{12} \\ 4\frac{S_3(n)}{3} &= \frac{(4n^2+4n+1)n(n+1) - n(n+1)}{12} \\ 4\frac{S_3(n)}{3} &= \frac{(4n^2+4n)n(n+1)}{12} \\ 4\frac{S_3(n)}{3} &= \frac{n^2(n+1)^2}{3} \\ S_3(n) &= \frac{n^2(n+1)^2}{4}. \end{aligned}$$

# 8 Solutions to Symmetry Problems

1. Given a standard $3 \times 3$ tic-tac-toe board, how many essentially different ways are there to make the first move?

   Intuitively, the answer is 3: the center square, a corner square or an edge square. The board is obviously symmetric under rotation by $90^\circ$ so every corner square can be converted to any other and similarly for the edge squares.

   What we are looking for is this: a mapping from the squares of one board to another such that if a series of moves is made on the first board then the same series of moves played on the images of those squares on the second board would be a valid game, and any winning or losing position on the first board would map to a winning/losing position on the second. In other words if you had a playing strategy on the first board, if you played the same strategy on the images of the squares on the second board, your strategy would work equally well.

   How do we know that there isn't some weird symmetry that basically preserves the properties of the tic-tac-toe game, but jumbles the squares around in some other way? (There *is* such a jumbling possible for the $4 \times 4 \times 4$ three-dimensional version of tic-tac-toe.)

   If the symmetric board is essentially the same, then the number of lines through each square must remain the same. The center square has four lines through it. In other words, you could possibly form a winning line in four different ways that included the center square. The corner squares are members of three possible winning lines and the edge squares, of two. Thus after any symmetric mapping of the board, the center, corner and edge squares must map to the center, corner and edge squares, respectively.

2. Can you arrange the numbers 1 through 9 on a $3 \times 3$ grid so that every row, column and diagonal adds to the same number? How many ways are there to do this? (This is called a magic square.)

   If we add all the numbers in all three rows on the grid, the total must be the sum of the numbers from 1 to 9: $1 + 2 + \cdots + 9 = 45$. If the three rows add to the same thing, each must add to $1/3$ of that, or 15.

   Assuming that we label the squares in the grid as follows:

   | | | |
   |---|---|---|
   | a | b | c |
   | d | e | f |
   | g | h | i |

   we know that:

   $$\begin{aligned} 15 &= a + e + i \\ 15 &= b + e + h \\ 15 &= c + e + g \\ 15 &= f + e + d \end{aligned}$$

   Add the equations above together to obtain:

   $$(a + b + c + d + e + f + g + h + i) + 3e = 60. \tag{1}$$

   Just adding together three rows or three columns gives us:

   $$(a + b + c + d + e + f + g + h + i) = 45. \tag{2}$$

   If we subtract Equation 2 from 1 we obtain $3e = 15$, or $e = 5$, so the center location in this $3 \times 3$ magic square must be 5.

   Now, with a 5 in the center, where can the 9 go? If you try to place it in a corner, a 1 goes in the opposite corner and you will find that it is impossible to fill in the other two corners. Thus 9 goes on an edge (with 1 opposite). The only possible numbers that can go in the row where the 9 sits are 4 and 2. Once these are filled in, the only possible board is the following:

   | | | |
   |---|---|---|
   | 4 | 3 | 8 |
   | 9 | 5 | 1 |
   | 2 | 7 | 6 |

   plus every possible rotation or reflection of the grid, for a total of 8 possibilities that are all essentially the same.

3. A game is played between two players. Begin with the numbers 1 through 9 written on the board. Players take turns selecting numbers from the board, and each time a number is selected, it is added to the player's pile and erased from the board. If a player can obtain a set of three numbers that add to 15 he/she wins. What is a good strategy for this game?

   A little reflection will show that this game is exactly the same as playing tic-tac-toe on the grid above. Selecting numbers is the same as putting your "X" or "O" on a square. Picking 5 is equivalent to picking the middle square, et cetera.

4. (*) Suppose that tic-tac-toe were played on a $4 \times 4$ board, and the goal is to get four squares in a row. How many symmetries would such a board have?

   As with the usual $3 \times 3$ tic-tac-toe board we will have the same rotation and mirroring possibilities on this larger board. There is a difference, however, since there is no longer a single center square, but now there are four of them. To keep track of what's going on, let's label the 16 squares on the original board as follows:

| | | | |
|---|---|---|---|
| A | B | C | D |
| E | F | G | H |
| I | J | K | L |
| M | N | O | P |

   Squares $A, D, M, P, F, G, J$ and $K$ all have exactly three lines passing through them. Similarly, squares $B, C, E, H, I, L, N$ and $O$ have two lines passing through them. There are two kinds of lines, too: lines containing only squares with three lines passing through them, and lines with two of each kind of square. The diagonals are of the first type; all the horizontal and vertical lines are of the second type. Any mapping between boards must map lines of the first type onto lines of the first type, and vice-versa.

   For example, the points $A$ and $D$ could not both map to the same diagonal line. This means that $A, D, M$ and $P$ must all map to corner squares, or all map to center squares.

   Could we map all the corner squares to center squares and vice-versa? In other words, could be build a valid mapping beginning with something like this?

| | | | |
|---|---|---|---|
| F | | | G |
| | A | D | |
| | M | P | |
| J | | | K |

   The answer, perhaps somewhat surprisingly, is yes. You can complete the square above mechanically as follows. To fill in the second square on the top row, note that it must lie on both the lines $FG$ and $AM$. In the original square, this can only be the square $E$. Continue in this way to obtain the following mapping:

| | | | |
|---|---|---|---|
| F | E | H | G |
| B | A | D | C |
| N | M | P | O |
| J | I | L | K |

   There are only 18 distinct lines in the original square, so it does not take too much time to verify that in the image square above that every one of the 18 lines maps to a unique line in the image. Thus we have a line-preserving map.

   Is that all? It turns out there is another sort of mapping that works. Imagine leaving the corners where they are, but swapping around the squares in the center:

| | | | |
|---|---|---|---|
| A | C | B | D |
| I | K | J | L |
| E | G | F | H |
| M | O | N | P |

   Remember that all the operations we've found can be combined, so there's another map that leaves the inner four squares where they are and swaps around the corner squares. This can be obtained by combining this mapping with the one we looked at previously.

In total, there are 4 rotations (including the rotation by $0°$), two reflection choices (reflect or not), and $2\times2$ ways to map inner to outer squares (including doing nothing). Thus there are $4\times2\times2\times2 = 32$ line-preserving mappings of the $4 \times 4$ tic-tac-toe board.

5. (**) Given a $4\times4\times4$ three-dimensional tic-tac-toe board, how many essentially different ways are there to make the first move? (Hint: there is an "obvious" answer, but the real answer is surprising and amazing.)

   There is a set of obvious symmetries of the rigid $4 \times 4 \times 4$ cube into itself. There are 24 of these. An easy way to see this is as follows: a particular corner can be moved to any one of the other 8 corners, but once it is placed there, it can be rotated to any of 3 positions (since there are three equivalent edges coming out of each corner). Since $3 \times 8 = 24$, there are 24 rigid motions that take the cube to itself.

   These operations, of course, leave the cube orientation as it was; it is also possible to reflect the cube, and this doubles the number of "obvious" symmetries to $24 \times 2 = 48$. The question is, "Is that all?"

   We can try to prove that we've got all of them with the 48 obvious ones in the same way we approached the $3 \times 3$ tic-tac-toe board. For each of the $4 \times 4 \times 4$ cubes in the larger cube, we can count the number of 4-long lines passing through it. In the $3 \times 3$ case, there were three different counts, so squares with equivalent counts had to map to squares with similar counts. What is the situation with a $4 \times 4 \times 4$ cube?

   A little reflection shows that there are (apparently) 4 different types of small cubes that make up the larger one. Those are: the 8 center cubes, the 8 corner cubes, 24 cubes in the center of the outer faces and the 24 cubes that make up the edge cubes that do not lie on the corners.

   Each corner cube has 7 lines passing through it: the edges of the large cube, the diagonals on the three adjacent faces of the large cube, and the long diagonal that passes through the center of the large cube.

   Each center cube also is part of 7 lines: the three lines perpendicular to the faces of the large cube, the diagonals on the three slices that are parallel to the outer faces of the large cube, and one long diagonal.

   In a similar manner we find that both the non-corner face cubes and the non-corner edge cubes each have 4 lines passing through them, so all we can conclude is that if there is another line-preserving map of the large cube to itself, the corners need to go to corners or center cubes and the non-corner, non-center cubes need to map to themselves.

   There are two kinds of lines: the long diagonals that pass from a corner of a cube to the opposite corner, passing through the small center cubes and each of the small cubes in such a line has 7 lines passing through it. All the other lines contain two of the 7-line small cubes and two of the 4-line small cubes. For this reason, The long diagonals (there are four of them) always have to map to long diagonals.

   If there is another symmetry, an obvious one would be to map each corner to its adjacent center cube and vice-versa. Here is an illustration of the four slices through the cube with the left-to-right blocks representing the four slices. The numbers in the slices represent the $x$, $y$ and $z$ "coordinates" of the small blocks. The first row represents the original contents of the small cubes and the second, the images of the corners and centers only after this inversion (the empty slots are not yet determined):

| | | | |
|---|---|---|---|
| 111 | 112 | 113 | 114 |
| 121 | 122 | 123 | 124 |
| 131 | 132 | 133 | 134 |
| 141 | 142 | 143 | 144 |

| | | | |
|---|---|---|---|
| 211 | 212 | 213 | 214 |
| 221 | 222 | 223 | 224 |
| 231 | 232 | 233 | 234 |
| 241 | 242 | 243 | 244 |

| | | | |
|---|---|---|---|
| 311 | 312 | 313 | 314 |
| 321 | 322 | 323 | 324 |
| 331 | 332 | 333 | 334 |
| 341 | 342 | 343 | 344 |

| | | | |
|---|---|---|---|
| 411 | 412 | 413 | 414 |
| 421 | 422 | 423 | 424 |
| 431 | 432 | 433 | 434 |
| 441 | 442 | 443 | 444 |

| | | | |
|---|---|---|---|
| 222 | | | 223 |
| | | | |
| | | | |
| 232 | | | 233 |

| | | | |
|---|---|---|---|
| | | | |
| | 111 | 114 | |
| | 141 | 144 | |
| | | | |

| | | | |
|---|---|---|---|
| | | | |
| | 411 | 414 | |
| | 441 | 444 | |
| | | | |

| | | | |
|---|---|---|---|
| 322 | | | 323 |
| | | | |
| | | | |
| 332 | | | 333 |

   If we begin that way, we can complete the entire board, since every cell that is not a center or a corner cell is at the intersection of two 3-dimensional lines that contain two center and two corner cells. For example, the point 112 lies on the intersection of lines $11x$ and $xx2$ in the original diagram, so it must also in the final one. (In this example, let $x$ vary from 1 to 4 to get all the elements on the line: 111, 112, 113, 114 for the first line and 112, 222, 332, 442 for the other.)

| | | | |
|---|---|---|---|
| 222 | 221 | 224 | 223 |
| 212 | 211 | 214 | 213 |
| 242 | 241 | 244 | 243 |
| 232 | 231 | 234 | 233 |

| | | | |
|---|---|---|---|
| 122 | 121 | 124 | 123 |
| 112 | 111 | 114 | 113 |
| 142 | 141 | 144 | 143 |
| 132 | 131 | 134 | 133 |

| | | | |
|---|---|---|---|
| 422 | 421 | 424 | 423 |
| 412 | 411 | 414 | 413 |
| 442 | 441 | 444 | 443 |
| 432 | 431 | 434 | 433 |

| | | | |
|---|---|---|---|
| 322 | 321 | 324 | 323 |
| 312 | 311 | 314 | 313 |
| 342 | 341 | 344 | 343 |
| 332 | 331 | 334 | 333 |

It is a little tedious to check, but if you do, you will find that every line on the original board maps to a line on the board above; it is essentially turned inside-out.

Finally, it is clear that if one corner goes to a corner then all the corners go to corners and if one corner goes to a center, then all the corners go to centers. This is because the long diagonals have to map to other long diagonals. However, there is nothing that says the order of the elements on the long diagonals has to be the same. In fact, if we leave the outer corners in place and invert all the centers as below, that can be extended to a line-preserving mapping:

| | | | |
|---|---|---|---|
| 111 | 113 | 112 | 114 |
| 131 | 133 | 132 | 134 |
| 121 | 123 | 122 | 124 |
| 141 | 143 | 142 | 144 |

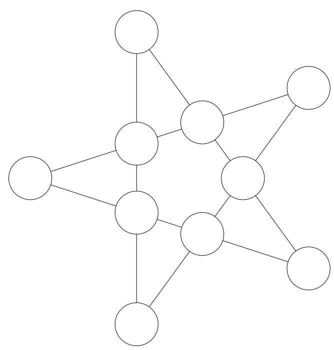
| | | | |
|---|---|---|---|
| 311 | 313 | 312 | 314 |
| 331 | 333 | 332 | 334 |
| 321 | 323 | 322 | 324 |
| 341 | 343 | 342 | 344 |

| | | | |
|---|---|---|---|
| 211 | 213 | 212 | 214 |
| 231 | 233 | 232 | 234 |
| 221 | 223 | 222 | 224 |
| 241 | 243 | 242 | 244 |

| | | | |
|---|---|---|---|
| 411 | 413 | 412 | 414 |
| 431 | 433 | 432 | 434 |
| 421 | 423 | 422 | 424 |
| 441 | 443 | 442 | 444 |

Of course both of these two non-obvious symmetries can be combined with any of the others, making a grand total of $48 \times 2 \times 2 = 192$ line-preserving symmetries.

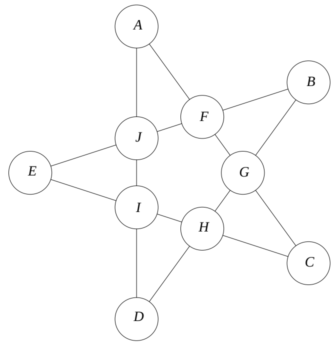
6. Given a pentagram with 10 holes as in the figure below, fill in the holes with the following 10 numbers: $1, 2, 3, 4, 5, 6, 8, 9, 10$ and 12 such that the sum of the numbers on each of the ten straight lines is the same.



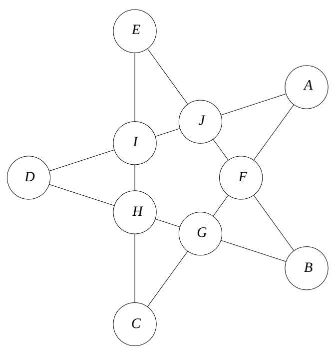
The first thing we need to do is figure out exactly what the sum is on each line. A standard trick is this: Imagine that we added together all the numbers on all the lines. Note that with this particular diagram, that summation would include every line exactly twice, since each hole appears on exactly two lines. The sum of all the numbers is 60, so adding all the numbers on all the lines would yield 120. Since there are five lines, each line must add to $120/5 = 24$.

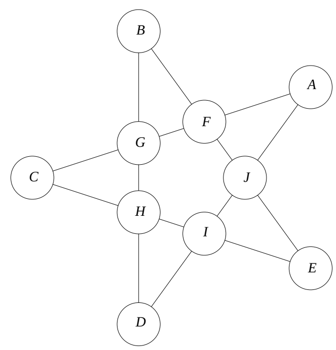
There doesn't seem to be many restrictions on what numbers can go together, but we do notice that no line can contain both the numbers 10 and 12, since together they add to 22, and the two smallest remaining numbers, 1 and 2 would still make the line total 25. Other pairs would make it even larger. You can list all the possible combinations of four numbers that add to 24, and there aren't too many, but it does turn out that other than 10 and 12, any other pair of numbers could possibly go on the same line. At least we know that once we place the 12, the position of the 10 will be restricted to only a few places (three, to be exact, no matter where the 12 is placed).

Another question arises, and that is, "Is the solution unique?" And, "What is meant by 'unique'?" If we find a solution that works, we could rotate the numbers one hole clockwise, and we'd have another. In fact, there are 5 possible rotations of a solution that are all basically the same. Another thing we could do would be to take our solution and mirror it so that if you read the solution numbers in a clockwise direction, the mirrored version would read them in a counter-clockwise direction. This could be combined with any rotation, giving us 10 solutions that are basically the same: they are just rotations or mirrorings of each other.



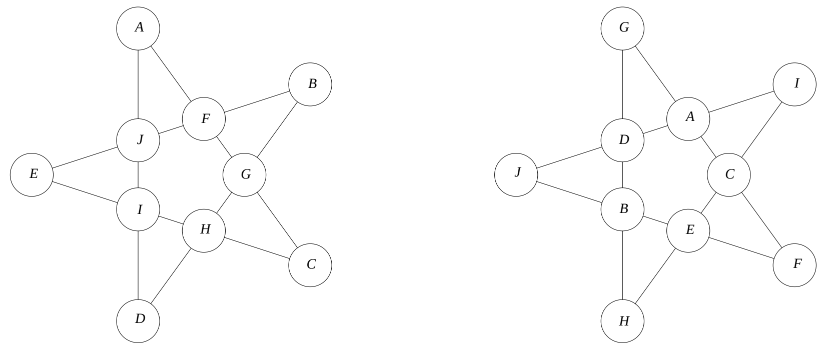

To make this concrete, imagine that the numbers $A$ through $J$ form a solution when arranged in a diagram as above. Then either of the following arrangements would also yeild a solution. The diagram on the left is rotated one position clockwise, and the one on the right mirrors the one on the left about the line $AH$.




The thing that makes the three figures above "equivalent" is the fact that if you imagine placing the actual numbers into the holes $A$ through $J$, if the $A$, $B$, $C$, ... numbers in one diagram are moved to the corresponding $A$, $B$, $C$, ... slots in another diagram, then they would either both be solutions or both not solutions. What is really going on is that four letters, say $AFGC$ in one diagram lie on a line, then they also form a line in both of the other lines. A mathematican might call this transformation a "line-preserving automorphism".

The usual rigid symmetry operations (rotation and reflection, in this case) will always preserve lines, but are there any others? In other words, can we map the letters from $A$ to $J$ to another figure so that all the lines are preserved but in a way that does not correspond to any of the ten mappings we've already discovered. (One of the ten mappings is just "don't change anything"; the others are either rotations, or a reflection followed by a rotation. The answer is "Yes!" and in ways that are quite surprising.

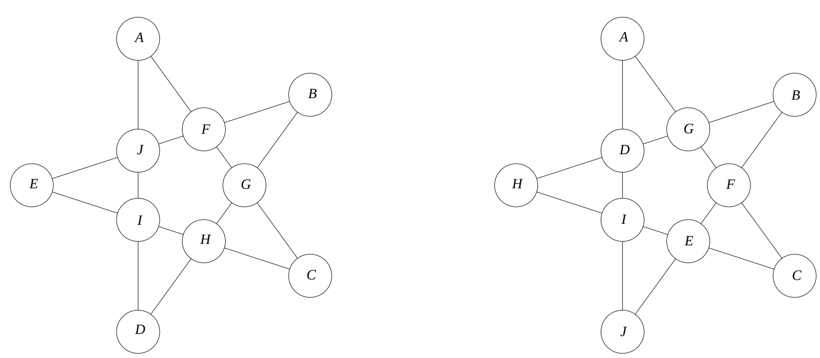



The figures above illustrate one such mapping. Assuming the diagram on the left is the initial labeling, one such line-preserving automorphism is shown to its right. Note that every number that was on the point of a star is mapped to the internal pentagon and vice-versa. To check that the lines are preserved, check that every four letters that form a line on the left continue to do so on the right. Since every line is mapped to a line, any assigment on numbers that forms a solution on the left will also form a solution in the diagram on the right. This transformation appears to turn the figure inside-out. Obviously, this transformation could be combined with any of the rotations and translations we already have to make a total of 19 additional equivalent solutions for any solution that we can find.

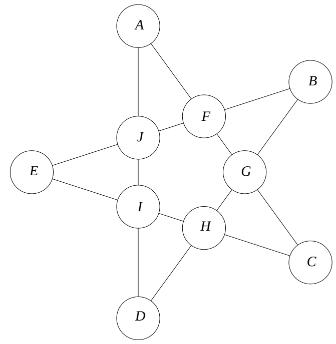
Is that all the possibilities? In every example we have so far, all the outer points go to outer points or else they all go to inner points. Are there any line-preserving automorphisms that leave some of the outer points on the outside, but move some of them to the middle?

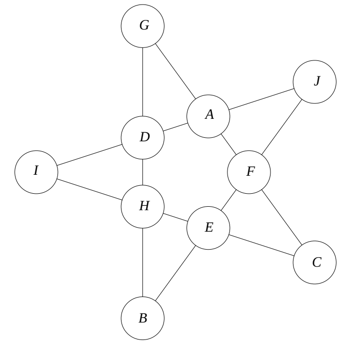
If at least one outside number stays on the outside and one moves inside, we may as well assume that the one outside is next to one that moves inside, so if $A$, for example, stays on the outside and $E$ moves inside, we may as well assume that $A$ stays in place (which we can arrange by a simple rotation, if necessary). Where can $E$ go? It is not on a line with $A$, so after moving inside, it must go to $H$, since $H$ is the only inside point that does not line on a line with $A$.

If $E$ goes to slot $H$, $B$ has to remain where it is or go to the $E$-slot, since it is not on a line with $A$ and the only positions not on a line through $A$ are $B$, $E$ and $H$. The slot $H$ is taken. But if $B$ goes to the $E$ slot, we could mirror that about the $AH$ axis, so we might as well leave $B$ where it is. Where does $H$ go? It's not on a line with $A$ and it is on a line with $B$, so it must go to the $E$ slot. With a little experimentation, we can find the following line-preserving automorphism from the figure on the left to the figure on the right:




Of course the automorphism above can be combined with any and all of the ones we found previously, and the surprising thing is that the grand total, after introducing the one above, is 120. This even includes some seemingly bizarre mappings that move the numbers in cycles of three. Here's an example below where $A$, $F$ and $G$ cycle as well as $E$, $H$ and $I$, and finally $B$, $D$ and $J$. Only the point $C$ remains fixed:




This means that if you can find any single solution to the puzzle, there are 119 others that are "equivalent", meaning that a simple remapping of the lines can convert any one to any other.

The number 120 is also interesting, since $5! = 120$, and since 5! is the number of rearrangements of 5 items, and finally, that the puzzle diagram contains exactly 5 lines. It turns out, of course, that any permutation of the lines will correspond to a specific permutation of the holes, and it's easy to figure out how to assign holes, given a line permutation: every pair of lines intersect in exactly one hole, so whatever letter was on both lines before the permutation will have to be on the unique intersection of those lines after the permutation. This is a much cleaner way to arrive at the magical 120.

Now, all we need to do is find one solution and we will effectively have 120 of them.

It turns out that there are not too many possibilities for sets of four numbers that make up lines that add to 24. Here is a complete list:

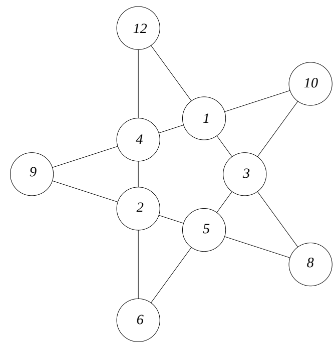
| | | | | |
|---|---|---|---|---|
| $a$ | 12 | 9 | 2 | 1 |
| $b$ | 12 | 8 | 3 | 1 |
| $c$ | 12 | 6 | 5 | 1 |
| $d$ | 12 | 6 | 4 | 2 |
| $e$ | 12 | 5 | 4 | 3 |
| $f$ | 10 | 9 | 4 | 1 |
| $g$ | 10 | 9 | 3 | 2 |
| $h$ | 10 | 8 | 5 | 1 |
| $i$ | 10 | 8 | 4 | 2 |
| $j$ | 10 | 6 | 5 | 3 |
| $k$ | 9 | 8 | 6 | 1 |
| $l$ | 9 | 8 | 5 | 2 |
| $m$ | 9 | 8 | 4 | 3 |
| $n$ | 9 | 6 | 5 | 4 |

Remember that in any solution, there have to be exactly two lines that pass through each number, and those lines cannot share any other numbers. If we ask for pairs of lines that pass through 12, there are only three possibilities: $ae$, $bd$ and $ce$.

The number 10 cannot be on a line with 12, so for each of the possible pairs of lines above containing 12 we need to find two lines containing 10 that have exactly one number in common with the two lines containing 12.

For $ae$ the only possibilities are $h$ and $i$. For $bd$, the only possibilities are $f$, $g$ and $j$. For $ce$ there is only one possibility: $f$. So $ce$ is impossible, and the only possibilities for the four lines are: $aehi$, $bdfg$, $bdfj$ and $bdgj$. Since the fifth line has to intersect all these four at exactly one point, just a tiny bit of work shows that the fifth line (which must be chosen from amoung $k, l, m$ and $n$) must be $l$ and the only valid line combination is $bdfjl$.

Since every permutation of the lines yields a valid result, assign them at random to the lines in the star and fill in the numbers. For example, to find the number that goes in the intersection of lines $b$ and $l$, since $8$ is the unique number that is on both lines, $8$ goes in that slot. Every permutation of the lines gives a different solution, and there are exactly $120$ of them. Here is one of the $120$ solutions:




7. Suppose you decide to sell sudoku puzzles to your local newspaper, but you are too lazy to work out any actual puzzles, so your plan is to steal an existing puzzle and modify it so that it is not easily recognized. What operations can be applied to an existing puzzle so that the resulting puzzle looks different? (Hint: one very easy idea is to place a $2$ whereever the original puzzle had a $1$ and vice-versa.)

   As was implied in the hint, the fact that the sudoku symbols are numbers is irrelevant; no calculations are done with them, and as long as you have $9$ different symbols, they would work just as well (although it's easier for a human to see a missing number than a missing arbitrary symbol).

   So the numbers can be rearranged in any way whatsoever and a new Sudoku puzzle will be generated. There are $9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362880$ rearrangements (a mathematician would say "permutations") of the nine numbers. To see why, we can choose any of the nine digits to replace the $1$. For each of those $9$ choices, there remain $8$ choices for the digit to replace $2$, for $9 \times 8$ total, et cetera.

   The board can also be rotated and/or mirrored entirely.

   If we call the smaller $3 \times 3$ units in a puzzle "blocks", then the puzzle can be viewed as three rows of three blocks, and those rows can be arbitrarily permuted. Or we could view it as three columns of three blocks which can also be arbitrarily permuted.

   Finally, within each row of blocks the three $1 \times 9$ rows can be permuted, and similarly for the $9 \times 1$ columns within each column of three blocks.

   All the above can be arbitrarily applied to a given puzzle to construct one that is in a sense, mathematically identical, but to a human, completely unrecognizable as a different version of the same puzzle.

8. If there are $270725$ ways to choose four cards from a deck of $52$, how many ways are there to choose $48$ cards from a deck of $52$?

   One way to specify which $48$ cards are chosen is to simply list the $4$ missing cards. Thus to every set of $4$ missing cards corresponds exactly one choice of the $48$ cards. Thus there are the same number of sets: $270725$.

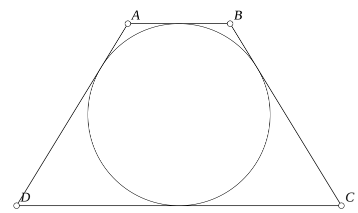
   If you're familiar with the equation used to count combinations, the equation itself is symmetric. The number of ways to choose $k$ things from a set of $n$ things, is called "$n$ choose $k$", and is usually written as follows, and has the following value:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

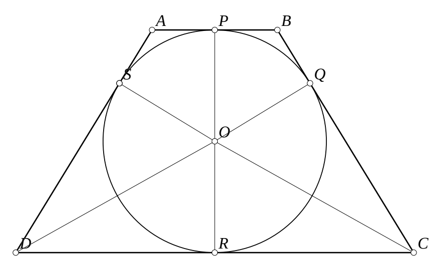
If we replace $k$ by $n - k$ in the formula, we obtain:

$$\frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

9. A circle is inscribed in an isosceles trapezoid as in Figure 3. (An isosceles trapezoid is a trapezoid where the two non-parallel sides ha ve equal length. In the figure below, the trapezoid is isosceles if $AD = BC$.) If segment $AB$ has length $l$ and segment $CD$ has length $L$, how long are the other two sides, $BC$ and $DA$?




Drop perpendiculars from the center of the circle $O$ to each of the sides, meeting them at points $P$, $Q$, $R$ and $S$ as in the figure below, and connect $O$ with all the identified points on the trapezoid:




It is obvious from the figure that $BP = BQ$, that $CQ = CR$, and that $BP$ and $CR$ are half the lengths of the top and bottom of the trapezoid. Thus $BC = (AB + CD)/2$ and by similar reasoning, $BC = AD$.

10. Given the following system of two equations and two unknowns, where the numbers $a, b, c, d, e$ and $f$ are constant:

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

Suppose an oracle tells you that for any (well, *almost* any[3]) set of values for $a, b, c, d, e$ and $f$ that the solution for $x$ is given by:

$$x = \frac{ce - bf}{ae - bd}.$$

How can you find the value of $y$, with minimal effort?

[3]To be precise, for any values such that $ae - bd \neq 0$.

If you simply rewrite the equations like this:

$$\begin{aligned} by + ax &= c \\ ey + dx &= f \end{aligned}$$

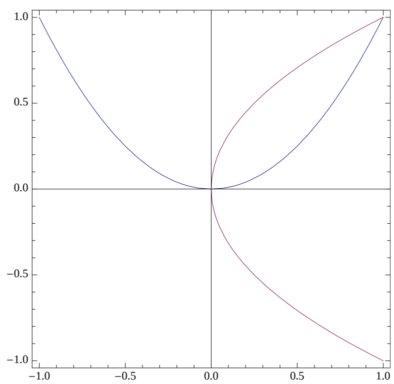
then they look exactly like the first set with $y$ in $x$'s position and vice-versa. Thus if we swap $a$ and $b$ and at the same time $d$ and $e$ in our solution, we will have an equation for $y$:

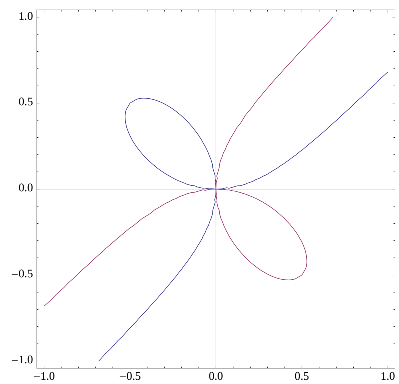
$$y = \frac{cd - af}{bd - ae}.$$

11. What is the relationship between the following pairs of graphs: $y = x^2$ and $x = y^2$? How about $x^3 - y^3 - xy = 0$ and $y^3 - x^3 - yx = 0$?

    In both of the cases, the two relations shown are identical in the sense that if you replace $x$ by $y$ and vice versa in one, you obtain the other. If you draw the graphs, essentially all you need to do is to swap the $x$ and $y$ axes (moving positive $x$ to positive $y$ and vice-versa). This is equivalent to mirroring the graph across the line $x = y$ which splits the positive $x$ and $y$ axes at a $45°$ angle.

    Shown below are the two graphs: the one on the left shows (in different colors) the graphs of $y = x^2$ and $x = y^2$ and the one on the right, in the same way, the graphs of $x^3 - y^3 - xy = 0$ and $y^3 - x^3 - yx = 0$.

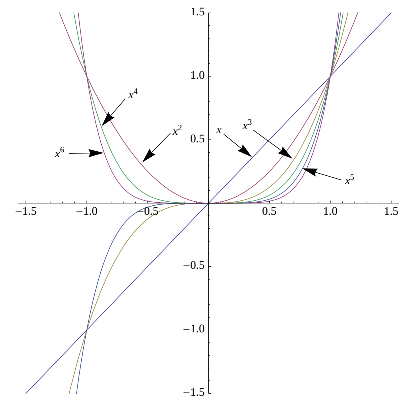





12. What sorts of symmetries can you find in the graphs of the following equations. For example, which ones will be symmetric about the $x$-axis, the $y$-axis, et cetera. What other symmetries can you find?

    - $y = x^2$.
    - $y = x^3$.
    - $y = x^n$, where $n$ is a positive integer.
    - $x^2 + y^2 = 25$.
    - $x^2 + 3y^2 = 25$.
    - $y = 1/x$.
    - $x^2 - y^2 = 1$.

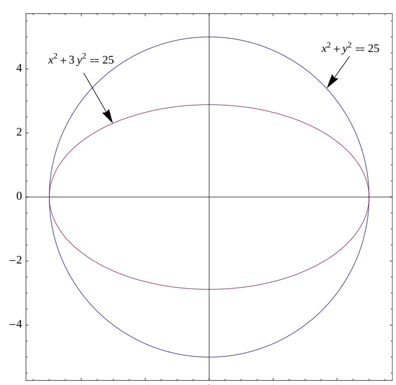
    The graphic below shows the graphs of the functions: $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4$, $f(x) = x^5$ and $f(x) = x^6$. Those with even powers of $x$ are symmetric about the $y$-axis and that is because $f(x) = f(-x)$. It is obvious mathematically that $f(x) = f(-x)$ just by plugging $-x$ in for $x$ in the function definitions. But the symmetry occurs because the height of the curve above $x$ and $-x$ has to be the same.

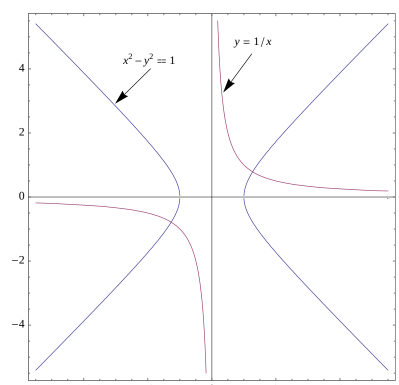



The functions with odd powers have a different kind of symmetry. Notice that in every case with odd powers we have: $f(-x) = -f(x)$. In other words, if you find the curve a certain distance above the line over the point $x$, it has to be that same distance below the line under the point $-x$ and vice-versa. There are two equally-valid ways to think about this type of symmetry. The first is this: if an odd function is mirrored first across one axis and then across the other, it lies exactly on top of itself. The second way of viewing it is that if you start at any point on the curve and reflect it across the origin, you will find another point on the curve. Or, more generally, if the curve is reflected through the point at the origin, it is unchanged.

Functions such that $f(x) = f(-x)$ are called "even functions" and for the same reasoning as above, when reflected across the $y$-axis they are unchanged. Functions such that $f(-x) = -f(x)$ are called "odd functions", and they remain unchanged after being reflected across both axes, or, equivalently, reflected through the point at the origin.

The next two graphs, $x^2 + y^2 = 25$ and $x^2 + 3y^2 = 25$, are symmetric in both $x$ and $y$, since replacing $x$ with $-x$ and/or $y$ with $-y$ leaves them the same. That means that the graph of either can be reflected across either the $x$-axis or the $y$-axis (or both) and the curve remains the same. Since it can be reflected across both axes with no change, as we noticed in the previous example, that means that an equivalent way of viewing the symmetry is that the curves can be reflected through the point at the origin and they will remain the same after reflection.



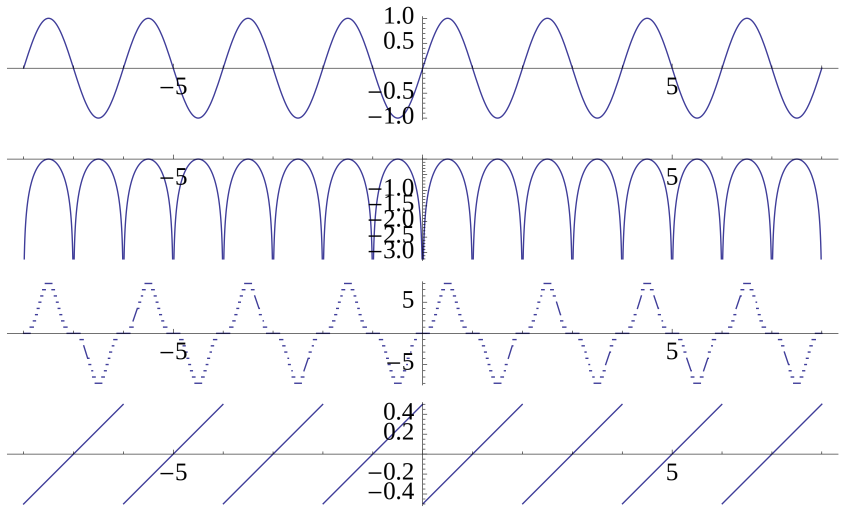

Finally, the graphic on the right above contains plots of $y = 1/x$ and $x^2 - y^2 = 1$. By the same reasoning as above, the first is an odd function and the second is symmetric in both $x$ and $y$.

13. What can you say about the graph of a function $f$ that satisfies the following conditions:

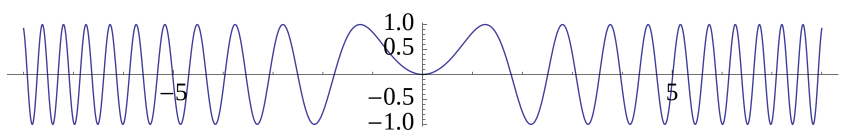
- What if $f(x) = f(-x)$, for all $x$?
- What if $f(x) = -f(x)$, for all $x$?
- What if $f(x) = f(x + 2)$, for all $x$?
- If $f$ is *any* function, what can be said about the graph of the function $f(x^2)$?

The first two questions were answered in the previous section and they are the even and odd functions.

The third is quite interesting: it says that if you shift two units to the right on the $x$-axis, the curve looks the same. This is known as a "periodic function" (in this case, with period 2). Illustrated below are a few examples of some functions with period 2. Note that they don't even have to be continuous, as in the last two examples:




The function $g(x) = f(x^2)$ is always even, since $g(x) = g(-x)$. The value of $x^2$ is always positive, so only values of $f$ for positive inputs are used, and they are reflected across the $y$-axis. The value of $x^2$ increases faster and faster as $x$ increases, so the graph of $f(x^2)$ will appear more and more compressed as $x$ increases. Shown below is the function $\sin x^2$. The function $\sin x$ is a nice periodic function that repeats forever in both directions will all its waves being the same size and shape. The graph below shows how $\sin x^2$ has similar waves, but reflected across the $y$-axis and that become more and more compressed as $x$ increases in absolute value.




14. There are two piles of coins on a table, each of which originally contains 10 coins. A game is played by two people who alternately select a pile and remove some number of coins (at least one) from that pile. The player who removes the last coin from the table wins. Does the first or second player have a winning strategy?

    The second player can always win. Whatever the first player does to one pile, the second player does to the other. That means the piles, after the second player has played, will continue to have equal numbers of coins. Thus when the first player finally empties a pile, the second player can do the same to the other pile and win.

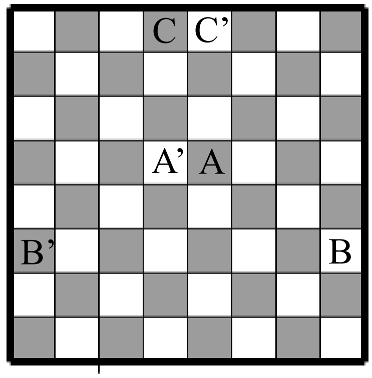
15. Consider the following game. Begin with an empty circular (or rectangular) table. Players alternate moves, and when it is your turn to move, you must place a quarter flat on the table. If there is no space left to do so, you lose. Does the first or second player have a winning strategy?

    The first player can always win. Place a coin at the exact center of the table, and then, after each play by the second player, exactly mirror that move on the other side of the central coin. If the second player can make a move then by symmetry, so can the first player after him.

16. Two players take turns placing bishops on a standard $8 \times 8$ chessboard, but once a bishop is placed, it is not moved, and no bishop can be placed on a square which is attacked by a bishop already placed. The first person who is unable to place a bishop on the board loses. Which player has a

winning strategy? (A bishop attacks all the squares that can be reached from its current square by moving along a diagonal in any direction.)

The second player always wins if he/she plays properly. Simply mirror the first person's move across the vertical line through the middle of the board (the horizonal middle line also works). In the figure below, if the first person makes moves $A$, $B$ and $C$, the corresponding moves by the second person should be $A'$, $B'$ and $C'$. Do you see why this always works?




17. A master chess player agrees to play against two novices as follows. The master will play the white pieces on one board and black on the other. He will make his move on the white board, then his opponent on the other board will make his first move with white. From then on, the novices will wait until the master responds, and then one of the novices will make a move. How can the novices assure one win and one loss or two draws?

    Each player simply copies the master's move on their chessboard, so the master is effectively playing against himself. The best he can hope for is to achieve one win and one loss, or two draws.

    (The author of this article once observed exactly this happening in a chess club. Since the master was going to lose a lot of rating points if the game was played out to the end, he started making bad moves, hoping that one of the opponents would try to win from a position where he had an obvious advantage and the master hoped that his chess was good enough to win, even though his opponent had a temporary advantage. Of course since the moves weren't terrible, his opponents copied him. So the master made worse and worse moves and finally, the move was deemed so bad that the opponent did not follow suit. After this, there were two different games being played, but with the master at a horrible disadvantage in both games. The master finally lost both of them!)

18. If you flip a fair coin $123$ times, at the end are you more likely to have more heads or more tails?

    It is exactly equally likely. Every sequence of $123$ flips has a matching one where the head and tail results are reversed. Thus for every sequence ending with more heads, there is exactly one ending with more tails.

19. An urn contains $500$ red balls and $400$ blue balls. Without looking at them, $257$ balls are removed from the urn and discarded. Finally, a single ball is drawn from the urn. What is the probability that it is red?

    The probability is $5/9$. Of course if you had looked at the colors of the discarded balls, you would have more information and a different probability.

20. Find the area under the curve $\cos^2 x$ from $x = 0$ to $x = \pi/2$.

    In the interval from $0$ to $\pi/2$ the cosine curve goes from $1$ to $0$ at the same time that the sine curve goes from $0$ to $1$ with exactly the same shape. Thus the area under the curve $\sin^2 x$ on that interval will be the same as the desired area. But if you add the two: $\sin^2 x + \cos^2 x = 1$, the area of both together will be equal to $1 \cdot \pi/2 = \pi/2$, so the desired area is half of that, or $\pi/4$.

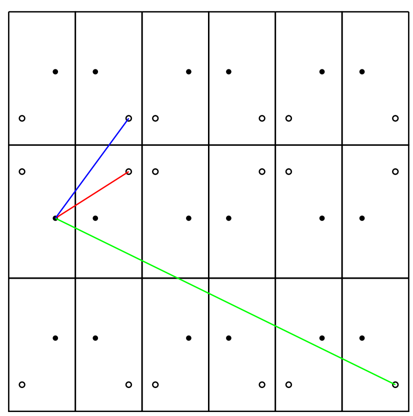
21. You have a cup of coffee and an identical cup of cream. Both contain the same amount of liquid. You take a tablespoon of cream and put it in the coffee. It is then mixed thoroughly and a tablespoon of the resulting mixture is added back to the cream. Is there now more cream in the coffee or more coffee in the cream? What if you don't mix thoroughly before you return the tablespoon of mixture to the coffee cup?

    Since at the end, both cups have the same amount of liquid, then whatever amount $x$ of coffee is in the cream cup must be exactly the same as the amount of cream in the coffee cup, namely, $x$.

22. A farmer with a bucket needs to water his horse. Both are on the same side of a canal that runs in a straight line. The farmer and his horse are on the same side of the canal, but the farmer needs to go to the river first to fill the bucket before he takes it to his horse. At what point on the canal should he collect the water to minimize the total distance he travels?

    Imagine mirroring the horse across the canal. No matter where the farmer fetches the water at the canal, it's just as far from there to the horse in the original position as it is to the horse in the reflected position. So if the horse were on the other side, the same path would minimize the route. Thus the farmer ought to go in a straight line from his position to the imaginary reflected horse and wherever that line hits the canal is the plance to go.

23. Suppose your cue ball on a normal rectangular billiard table is at point $P$ and the target ball is at point $Q$. Is it possible to hit the target after bouncing off one cushion? Two cushions? Three? How can you figure out which direction to hit the cue ball to achieve these results. (Assume that the cue ball does a "perfect" bounce each time, with the angle of incidence equal to the angle of reflection. Also assume that the table dimensions are exactly $2:1$.)

    See the figure above, and assume that the cue ball is the solid black circle and the target ball is the open circle. Draw reflections of the table over and over in every direction. On each reflected table, indicate where the target ball is. Select a path to the target ball that passes through the appropriate number of edges, and the ball will reflect off the edges in exactly the same way that the copies of the table reflect. In the figure above, the red path will hit the target after one bounce, the blue path will hit the target after two, and the green path will hit the target after six bounces.

24. In a room with rectangular walls, floor and ceiling, if a spider is on one of the surfaces and the fly on another, what is the shortest path the spider can take to arrive at the fly, if the fly does not move? (The answer, of course, will depend on the dimensions of the room, and upon where the spider and the fly initially start. What we're searching for is a method to find the solution for any room size and any initial positions of the spider and the fly.)

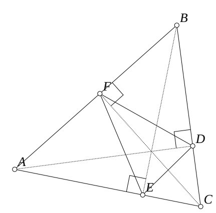
    Imagine the room as a cardboard box. The box can be cut and flattened in various ways (not too many) and the best path for the spider to the fly will be a straight line on one of the unfolded, flattened versions.

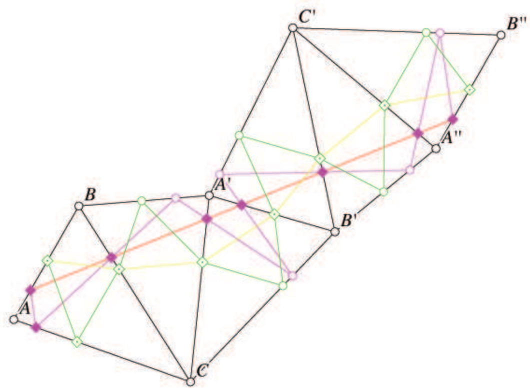
25. (*) If you build an elliptical pool table and you strike a ball so that it passes through one of the ellipse's foci, then after it bounces off a cushion, it will pass through the other focus. Show that this is true, based (loosely) on what you learned from the farmer and his horse a couple of problems ago. Remember that an ellipse is defined to be the set of all points such that the sum of their distances to the two foci is constant. Hint: what would the shape of a river be so that it doesn't matter where the farmer goes to get his water?

    You can turn the farmer/horse problem around and ask, given the positions of the farmer and the horse, and a certain distance the farmer is willing to walk, where are all the possible locations for the canal?

    If you draw all possible canals, they will form an envelope of an ellipse; namely, the points on the canals where the farmer gets the water will be points, the sum of whose distances to the farmer and horse, is constant. Thus the points form an ellipse.

26. (*) Fagnano's problem. Show that in any acute-angled triangle, the triangle of smallest perimeter that can be inscribed in it is the so-called "pedal triangle" whose vertices are at the feet of the altitudes of the given triangle. In the figure below, $\triangle DEF$ is the pedal triangle for $\triangle ABC$. What happens if the given triangle contains a right angle or an obtuse angle?

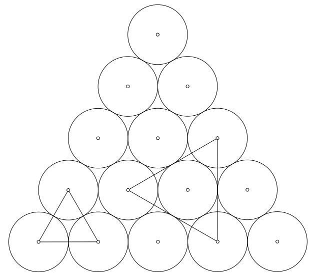


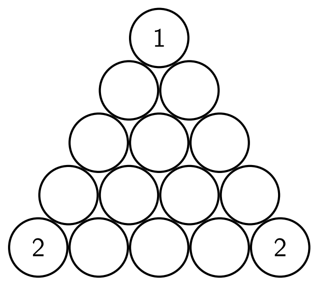

    Reflect the triangle 5 times over different edges as illustrated above and note that the red line that represents the pedal triangle forms a straight path across the reflections of the original triangle. The entire length makes up two complete perimeters of the pedal triangle.

    Any other triangle (for example the one in green) also forms a path that is twice the perimeter of the green triangle, but it is not a straight path, and therefore, is not the triangle of smallest perimeter. (Note that after all the reflections, the segments $AB$ and $A''B''$ are parallel.)

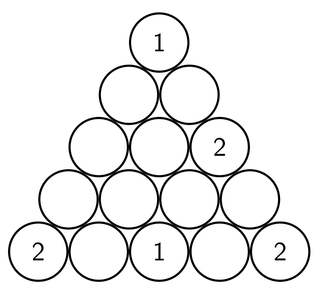
27. Fifteen pennies are placed in a triangular shape as shown below. Many sets of three centers of those pennies form the vertices of equilateral triangles, two samples of which are illustrated in the figure. Is it possible to arrange the pennies in such a manner that no set of penny centers that form an equilateral triangle are all heads or all tails?



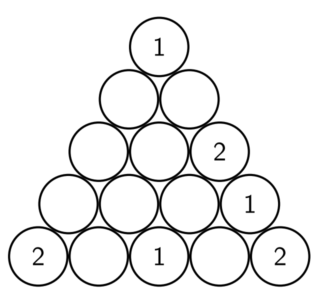
This problem can be solved by checking various possibilities, but symmetry makes the search much faster. For example, if there is a solution, then flipping all the coins over will be another solution. So we can just call the coins 1 and 2 and you can start, say by placing coins at the three vertices of the large triangle and labeling them 1, 2, and 2 (since they can't all be the same, and the 2s might as well be on the bottom).



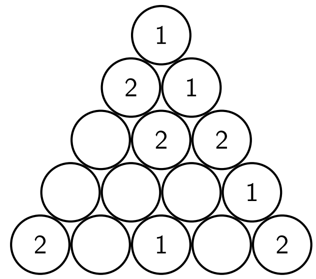

Then the coin half-way between the two 2s can't be a 2 or it would force two 1s in the third row making a triangle of 1s. Since everything is symmetric left-right, one of the coins on the end of the third row has to be a 2: put that on the right.



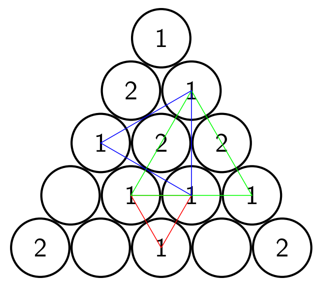

Reasoning as above, there has to be a 1 between the two 2's on the right edge:




Now there are three more forced moves:




Three more 1s are forced making three illegal triangles, so the required arrangement is impossible:




The series of illustrations above is harder than is needed, but this series of steps illustrates the use of symmetry many times. It is easier to construct the proof of impossibility by starting with the triangle completely enclosed in the center of the larger one and to label the vertices !, 2, and 2. Try it!

28. Add the whole numbers from 1 to 100.

    This is the same as adding the whole numbers from 100 down to 1, so if we add both of those sums (call the sum of each, $S$), and interleave the numbers, we obtain:

$$2S = 1 + 100 + 2 + 99 + 3 + 98 + \cdots + 100 + 1.$$

    Grouping, we obtain:

$$2S = (1 + 100) + (2 + 99) + (3 + 98) + \cdots (100 + 1).$$

    So twice $S$ is 100 copies of 101, so $2S = 10100$ and $S = 5050$.

29. Find the value of $x > 0$ which minimizes the function $f(x) = x + 1/x$.

    We want to minimize $x+1/x$ with $x$ positive. The same value of $x$ will also minimize the expression $x + 2 + 1/x$, since this expression is always exactly 2 more than the original. But:

$$x + 2 + 1/x = (\sqrt{x} + 1/\sqrt{x})^2,$$

    and to minimize the square of a function you need to minimize the function, in particular, $\sqrt{x} + 1/\sqrt{x}$. This is exactly the same as the original function but with $\sqrt{x}$ in place of $x$, so it will be minimized at the same point; namely, when $x = \sqrt{x}$, or when $x = 1$.

30. What is the area of the largest rectangle that can be inscribed in a circle of radius 1?

    When a rectangle is inscribed in a circle the line segment connecting a pair of opposite vertices is a diameter of the circle and it divides the rectangle into two identical triangles. If we consider that diameter as the base of the triangles, the area of both triangles can be maximized when the point opposite it is as far away as possible. This will occur at the halfway point, so the rectangle is a square with diagonal equal to 2. The lengths of the sides of the square are thus $\sqrt{2}$ so the area of the largest possible square is 2.

31. In a triangle with sides 1, 1 and $x$, find the value of $x$ that maximizes the area.

Imagine that one side of length 1 is the base of a triangle and rotate the other length-1 side through !80 degrees. The area starts at zero and ends at zero, and is maximized when the other end of the rotated side is as far from the base as possible. This occurs when the two sides make an angle of 90 degrees, so the third side must have length $\sqrt{2}$.

32. Give some strong evidence that an equilateral triangle is the triangle of largest area that can be inscribed in a circle. What is the largest quadrilateral that can be so inscribed? The largest $n$-sided figure?

Imagine that you have a fixed chord in the circle and you want to make the largest (in area) possible triangle by choosing the third vertex on the circle. The base (the chord) is fixed, so the area will depend only on the height which you would like to make as large as possible. This will occur where the perpendicular bisector of the chord intersects the circle on the side opposite the diameter from the chord, or if the chord is the diameter, then both intersection points are equally good.

At this point, the other two sides are equal.

Suppose there is a triangle with largest area that is not equilateral. Select any two sides that are not equal, and fix the third side as the chord. By moving the third point as described above, the area can be increased.

The same sort of argument applies to an arbitrary $n$-gon. If two adjacent sides are not equal, consider the chord that connects the points where they don't intersect and move the vertex where they intersect to a point that equalizes the lengths, and you'll have a larger area. Thus the largest possible area in an $n$-gon occurs when all the sides have equal length.

33. If you have a million distinct points inside a circle, can you find a line that divides them such that there are exactly half on each side?

For every pair of points, draw the line through them. This will make a lot of lines, but only a finite number. Pick a point in the circle that is not on any of the lines, and draw an arbitrary line through it. If that line separates the points in half, then you are done.

If not, suppose there are $x$ points on one side or your line and $y$ points on the other. Since they are not equal, the expression $x - y$ will not be zero. Now imagine that the line rotates and makes a 180-degree turn. Each time it crosses a point in the circle, either $x$ or $y$ is increased or decreased by 1. It cannot hit two points at once or the point you picked was on one of the lines connecting the pairs of the million points. Thus the value of $x - y$ changes by one (either up or down) each time the rotating line crosses a point.

But when the full 180-degree rotation is done, the value of $x - y$ will have the opposite sign, since the two sides of the line are exchanged. That means that at some point in between, the value of $x - y$ had to be zero.

34. Find all whole number values $a$, $b$ and $c$ such that $a + b + c = abc$.

By symmetry, any solution can generate another one with the values of $a$, $b$, and $c$ permuted, since all appear in the equation in the same way.

If one of the integers is zero then the product $abc$ is also zero, so we will have a solution with $a = 0$ and $b = -c$ which forces $b$ and $c$ to be zero.

We might as well assume that $a \leq b \leq c$, so $abc = a + b + c \leq 3c$, so unless all three numbers are zero, we have $ab \leq 3$. There aren't too many possibilities here, but one is $a = 1$, $b = 2$, and $c = 3$. (Plus, of course, the 5 other permutations of the answers.)

35. A cube is built with wire edges as in Figure 14. If wires are connected to opposite corners of the cube and a one-ampere current is passed through, how much current flows through each of the edges. (Not every edge will have the same current passing through it.)

By symmetry, the current flowing across the three wires connected to the input wire must be equal, so $1/3$ ampere must flow in each. Similarly for the three wires connected to the output wire. Again,

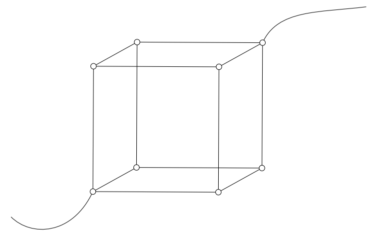


Figure 14: Wire cube

by symmetry, all of the wires that connect the nodes adjacent to the input and output wires all look the same. That means that an equal current flows through each, and since there are 6 of them, each has a current of $1/6$ ampere.

36. (*) A square metal plate has three sides held at a temperature of $100$ degrees and the fourth at zero degrees. What's the temperature at the point in the center of the plate?

    If the plate were completely surrounded by $100$ degree edges, clearly the center would be at $100$ degrees, so each contributes $25$ of them. If it were surrounded by 4 edges at $0$ degrees, the result would be $0$. The temperature at the center must then be $25$ degrees.

37. (*) An infinite square mesh of wire (a small part of which is shown in Figure 15) extends in every direction. All grid lengths are equal, and all the wire has the same resistance per unit length. Two wires are connected to adjacent grid points $A$ and $B$ and a one-ampere current enters through $A$ and leaves through $B$. What is the current through $AB$? Note that the electrons will follow many paths, with more following the shorter paths since the resistance is smaller.

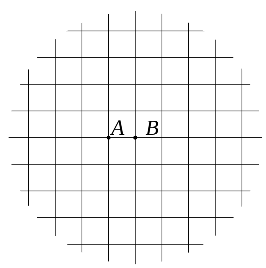



Figure 15: Inifinte wire mesh

This is solved using the idea of superposition and symmetry. Imagine first a circuit where the input wire is connected to $A$ but there is no output wire. As current flows into the infinite grid, the electrons go "out to infinity." By symmetry, $1/4$ ampere goes out along each of the four wires connected directly to $A$.

Now imagine the same thing, but with no input wire, but one ampere coming out from $B$. All the electrons flow "from infinity" and out along the $B$ wire. By symmetry, $1/4$ ampere comes in through each of the four wires connected to $B$.

By superimposing the two circuits, we have $1/4 + 1/4 = 1/2$ ampere flowing along the wire connecting $A$ to $B$. All the current that flows to and from "infinity" cancels, and the result is $1/2$ ampere.

(Note: the current is usually thought of as moving in the direction opposite the movement of the electrons, so the wording above is slightly misleading, at least to an electrical engineer.)

38. Evaluate the following three expressions using a (translation) symmetry observation:

$$\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$$

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

These are all solved in roughly the same way: give the result a name, and notice how that name appears again, hidden within the expression. After that, all that's needed is some algebra.

$$\begin{aligned} x &= \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} \\ x &= \sqrt{1+x} \\ x^2 &= 1+x \\ x^2-x-1 &= 0 \\ x &= \frac{1+\sqrt{5}}{2}. \end{aligned}$$

(The final line is obtained using the quadratic equation. Why do we only use the positive value of the square root?)

$$\begin{aligned} x &= \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} \\ x &= \frac{1}{1+x} \\ x^2+x &= 1 \\ x^2+x-1 &= 0 \\ x &= \frac{-1+\sqrt{5}}{2}. \end{aligned}$$

$$\begin{aligned} x &= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots \\ x/2 &= \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots \\ x-x/2 &= 1 \\ x/2 &= 1 \\ x &= 2. \end{aligned}$$

39. Solve for $x$:

$$2 = x^{x^{x^{x^{\cdots}}}}.$$

(*) Solve for $x$:

$$4 = x^{x^{x^{x^{\cdots}}}}.$$

What is going on here?

Since the exponents go forever, the exponent of the first $x$ is the same as the entire value, so for the first case we need to solve:

$$x^2 = 2$$

from which we obtain $x = \sqrt{2}$.

But if we do the same thing to the second equation, we obtain:

$$x^4 = 4$$

yielding $x = \sqrt{2}$.

Obviously, both can't be right. To see what is going on, we have to consider what might be meant by the infinite tower of exponents. For any given value of $x$ we might consider the following limit:

$$x, x^x, x^{x^x}, x^{x^{x^x}}, \ldots$$

We can do some experimental work easily by noting that if we have the numerical value for some point in the series, we just need to raise $x$ to that value to obtain the next. Let's try $x = \sqrt{2}$:

$$\begin{aligned} x &\approx 1.4142135623730950488 \\ x^x &\approx 1.6325269194381528448 \\ x^{x^x} &\approx 1.7608395558800280908 \\ x^{x^{x^x}} &\approx 1.8409108692910102984 \end{aligned}$$

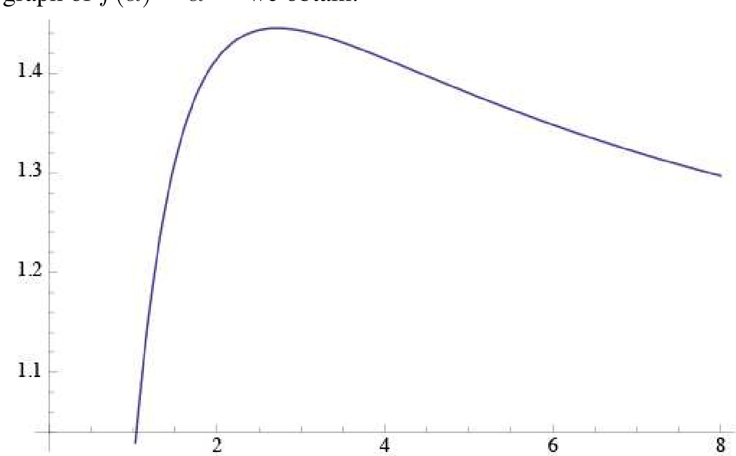
If we continue with the same numerical experiment 10 more times, we obtain a value of approximately $1.9945944507121011776$, which is pretty close to $2$. The exponents clearly get larger but if they are less than $2$ then raising $\sqrt{2}$ to that power will also be less than $2$, so it seems likely that $2$ is the limit, so $x = \sqrt{2}$ is the correct solution to the first problem.

To get the series to reach 4, we need the stack of exponents to be larger than 2, but once that happens the series diverges, so there is no answer to the second problem.

To see what's going on, let's replace the $2$ or the $4$ in the equations above by an arbitrary constant, say $\alpha$:

$$\begin{aligned} \alpha &= x^{x^{x^{x\cdots}}} \\ \log\alpha &= \alpha\log x \\ \log\alpha/\alpha &= \log x \\ \alpha^{1/\alpha} &= x. \end{aligned}$$

If we plot the graph of $f(\alpha) = \alpha^{1/\alpha}$ we obtain:



This function has a maximum at $\alpha = e = 2.718281828\ldots$ and in fact, the original equation with a tower of exponents can be solved if the constant in front is less than $e$ and otherwise, it cannot.

40. How quickly can you expand the following product?

$$(x + y)(y + z)(z + x)?$$

What is different about the product?

$$(x - y)(y - z)(z - x)?$$

In both examples, it is clear that the product will contain only terms with three variables in them, where a variable may be repeated. Ignoring duplicates, there will be 8 terms, since there are two choices in each pair of parentheses. If we want. There will be no terms like $x^3$ since none of the variables appear in all three pairs of parentheses. Thus all terms look like $\alpha^2\beta$ ($\alpha$ represents one variable and $\beta$ the other) or $xyz$. There are two ways to obtain $xyz$, leaving 6 more terms, and since everything is symmetric and there are only 6 ways to make terms of the form $\alpha^2\beta$, every one of those must appear once. So here is the solution to the first part:

$$(x + y)(y + z)(z + x) = 2xyz + x^2y + x^2z + y^2x + y^2z + z^2x + z^2y.$$

The second example is exactly the same, except that the signs will be different. In fact, one way of getting $xyz$ is by choosing the positive value in each pair of parentheses and the other way gives a negative value so there is no $xyz$ term in the final product. For the other 6 terms, half will be positive (choosing two negative terms and a positive one) and half negative (two positive terms and a negative one). Since each term has one variable squared and the other not, the squared variable comes from a combination of a positive and negative entry, so the result depends on the sign of the final variable in the third pair of parentheses.

Thus we obtain:

$$(x - y)(y - z)(z - x) = x^2x - x^2y + y^2x - y^2z + z^2y - z^2x.$$

41. If $\{x = 1, y = 2, z = 3\}$ is a solution for the following set of equations, find five more solutions. (*) Find all whole-number solutions for $x$, $y$ and $z$ in the equations below:

$$\begin{aligned} x + y + z &= 6 \\ x^2 + y^2 + z^2 &= 14 \\ xyz &= 6 \end{aligned}$$

Notice that since $x$, $y$, and $z$ appear symmetrically in the equations (in other words you can exchange any pair and obtain exactly the same set of equations), then for any single solution you find, all possible rearrangements of those values among $x$, $y$, and $z$ are valid solutions. Assuming that $x$, $y$, and $z$ have different values, for every particular solution you find, there will be 5 additional ones.

Since, $x = 1$, $y = 2$, and $z = 3$ is a solution, here are six solutions:

$$\begin{gathered} x = 1, y = 2, z = 3 \\ x = 1, y = 3, z = 2 \\ x = 2, y = 1, z = 3 \\ x = 2, y = 3, z = 1 \\ x = 3, y = 1, z = 2 \\ x = 3, y = 2, z = 1 \end{gathered}$$

To verify that these are the only possible solutions, it is easy to check every set of three numbers that multiply to give $6$ and there are not too many of those. In fact, this is the only solution, even if we are not restricted to whole numbers.

42. (**) Find all whole-number solutions for $w$, $x$, $y$ and $z$ in the following system of equations:

$$\begin{aligned} w + x + y + z &= 10 \\ w^2 + x^2 + y^2 + z^2 &= 30 \\ w^3 + x^3 + y^3 + z^3 &= 100 \\ wxyz &= 24 \end{aligned}$$

By similar reasoning as in the problem above, for every solution where $w$, $x$, $y$, and $z$ are different, all 24 permutations of the values will provide other solutions. In this case, one of the 24 is: $w = 1$, $x = 2$, $y = 3$, and $z = 4$.

To verify that these are the only possible solutions, it is easy to check every set of three numbers that multiply to give $24$ and there are more of those than in the previous problem, but not too many more. In fact, this is the only solution, even if we are not restricted to whole numbers.