# Solving BAMO Problems

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**Abstract**

Strategies for solving problems in the BAMO contest (the Bay Area Mathematical Olympiad). Only the first section is specific to BAMO; the rest of the document concerns general-purpose mathematical problem solving techniques.

## 1 BAMO

BAMO (the Bay Area Mathematical Olympiad) is an olympiad-style contest consisting of five problems to be solved in four hours. The term “olympiad style” means that each problem requires a written solution, generally in the form of a mathematical proof. The people who compose the exam try to arrange the problems roughly in order of difficulty, so most people should be able to solve problem 1, and almost nobody should be able to solve problem 5.

All problems are of equal value (7 points possible on each), and most of the scores on any particular problem will be 0, 1, 6, or 7; middle scores are rare.

Obviously you should try to work the problems in order; if you are having trouble solving problem 2, it’s unlikely that you will make much progress on problem 5. But of course you should look at all the problems. For example, if you are extremely good at geometry and the third problem is geometric, it may well be that problem 3 is easier for you than problem 2. Generally, however, the arrangement of the problems fairly accurately reflects their difficulty.

If you do solve a problem, rather than beginning work immediately on the next it is almost certainly a good idea to check over your solution and make certain it is clearly written, that you didn’t leave anything out, and that it in fact solves the problem you are trying to solve. It is a shame to get problems 2 and 3 correct and to get a zero on problem 1 since it seemed so easy that you made some silly mistake on it.

Remember that 4 hours is a long time, and it is often better to spend 5 extra minutes on problem 1 to make sure that it is perfect than to spend that 5 minutes making zero progress on problem 5.

## 2 Writing Solutions

Remember that there will be humans grading your work. They are trying to be as fair as possible, but if your writing is difficult to read, or the solution is disorganized or if the sentences are badly written and difficult to understand, you will make it hard for the reader to understand your solution, and will thus be less likely to get high marks. When you are finished with each problem, take a look at it and pretend that you are the person trying to grade it. How would you like to grade it?

Here are some ideas for how to write a proof or essay that is easy to understand:

1. First and foremost, remember everything you learned in your English writing classes. Organize your thoughts, use complete sentences, et cetera.

2. Write an outline before you begin, where "outline" simply means a sentence or two explaining how your proof works. For example, you might write something like this: "The proof will be done by induction on $n$, the number of sides of the polygon. We will show it is true for a triangle ($n = 3$), the smallest polygon and then we will induct on $n$. For $n$ larger than 3, the proof will be divided into two cases, depending on whether $n$ is odd or even." Then write your proof in three parts, ideally with a short header in front of each, like: "Case $n = 3$", "Case $n > 3$, $n$ odd", and finally, "Case $n > 3$, n even".

3. If you write something that you later decide you don't need or is incorrect, be sure to cross it out completely so that the reader will understand clearly that it is not part of your solution.

4. If your solution covers multiple pages, make sure you number them in an obvious way: "Problem 3, page 2", for example.

5. This was said above, but it is so important that it's worth repeating: when you finish writing a solution, take a few minutes to reread what you have written to make certain it says what you think it does.

# 3 How to Get Started

> Mathematics must be written into the mind, not read into it. "No head for mathematics" nearly always means "Will not use a pencil."
> *Arthur Latham Baker*

Do not spend a lot of time just staring at a blank sheet of paper. *Do something!* Try to search for a pattern, draw a picture, work out some simple cases, try to find a simpler related problem and work that, change the notation, et cetera.

Here are some ideas of activities you can usually do, even if you have no idea at all how to approach the problem:

1. **Search for a pattern:** Imagine you are asked to find the sum of the first few odd numbers:

$$S = 1 + 3 + 5 + \cdots + (2k + 1).$$

Work out the values for small values of k:

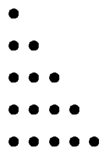
$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \\ 1+3+5+7+9 &= 25 \end{aligned}$$

All the answers are perfect squares! With a clue like this, it will probably be much easier to find out why.

2. **Draw a picture:** For a geometry problem this should be obvious, but you can often draw pictures for other problems as well. For example, suppose you want to show that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

   Here is a picture that might help:



   The sum is like a triangle, so is obviously related to the area of that triangle.

3. **Check some simple cases:** In the example above, check the first few values of $n$. When you check values, be sure to try the easiest ones first. In other words, don't check $n = 4$ until you've checked $n = 0$, $n = 1$, $n = 2$, and $n = 3$. Remember to try zero. If you're supposed to show something about a general triangle, try it on a few triangles that you can calculate with easily, such as an equilateral triangle or a $3 - 4 - 5$ right triangle.

4. **Solve a simpler related problem:** For example, if the problem asks about the arrangement of queens on a chessboard, try to solve the problem with boards that are smaller than an $8 \times 8$ chessboard: look at the $1 \times 1$ board, the $2 \times 2$ board, and so on.

5. **Change the notation:** If your problem involves, for example, binomial coefficients, replace them by the factorial equivalents:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

6. **Think about similar problems:** If this problem reminds you of one you have solved previously, how did you solve that one?

# 4 General Techniques

You can usually apply the techniques in the previous section even if you have no idea how to start. The techniques listed here are more specific, but it's worth keeping them all in mind when you approach a new problem. Remember that sometimes there are many techniques that will work; to get a top score, all you need to do is to find one of them.

1. **Divide into Cases.** If you know how to solve the problem under certain conditions, perhaps you can divide it into cases. Also, be certain to be sure that you have proved it in all cases; for example, if the problem concerns two parallel lines, be sure that your proof works if the lines are parallel. If it doesn't, you may have to prove that as a special case.

2. **Look for Symmetry.** Symmetry can be geometric or algebraic. For example, if you have to multiply out the term $(x+y+z)^5$, and after some struggle, you find that the coefficient of $x^2y^2z$ is 30, then so will be the coefficient of $x^2yz^2$ and of $xy^2z^2$ since the original expression was symmetric in $x$, $y$, and $z$.

3. **Use Induction.** If you can assign an integer "size" to each version of a problem and it looks like the problem for a larger size can be solved in terms of similar problems of smaller size, perhaps induction will work. Induction does not have to be used on algebraic problems. As an example, suppose you want to show that any polygon (convex or not) can be cut into triangles using diagonals that lie within the polygon. Surely the smallest polygon (a triangle) can be so divided. If you can then show that any polygon can be split into two smaller polygons with a diagonal, you can use induction to prove the desired result.

   The "Towers of Hanoi" problem is another good example.

4. **Work Backwards.** Lots of games work this way. For example, suppose you play a game where you begin with a pile of 50 sticks, and a move consists of taking 1, 2, 3, or 4 sticks from the pile. You alternate moves with your opponent, and the first person unable to make a move loses. If you move first, do you have a strategy that will guarantee a win?

   50 is a pretty big number, but work backwards. Who wins if the starting pile has zero sticks? 1 stick? 2 sticks? Work backwards to see which positions are "safe" to leave an opponent. Zero sticks is clearly safe, and piles with 1, 2, 3, or 4 sticks are unsafe. 5 is safe because any move your opponent makes leaves him in an unsafe position, and so on.

5. **Consider Parity.** Sometimes problems have an odd-even condition. Given a polygon with 101 sides that has an axis of symmetry, show that the axis passes through a vertex. This is easy if you pair each vertex with the symmetric vertex across the axis.

6. **Use the Pigeon Hole Principle.** If you place more than $n$ things into $n$ boxes, at least one box will have more than one thing in it. In a group of 13 or more students, at least two will have a birthday in the same month.

7. **Use Proof by Contradiction.** If you can't prove something, assume it is false and see what you can conclude from that. If you can conclude something that is obviously false beginning with that assumption, then your assumption must be wrong and therefore the original statement is true.

   Prove that there is no largest prime number. Assume there is a largest, say $P_n$, where $P_1, P_2, \ldots, P_n$ is the list of all the primes. Then multiply them all together and add 1: $X = P_1P_2\cdots P_n + 1$. $X$ can't be a multiple of any of the $P_i$, since if you divide $X$ by $P_i$ it leaves a remainder of 1. So either $X$ is prime or it is the product of primes not in the list. In either case the original assumption that there were only a finite number of primes leads to a nonsensical result, so there must be an infinite number of primes.

8. **Look for Invariants.** Sometimes these is a property of your problem that is preserved no matter what operations are performed. Here's a good example. Suppose you begin with a chocolate bar that is 8 squares by 5 squares and play the following game. If it is your turn to move, you select a piece (at the beginning, of course, you have only the original piece), and you break it along one of the lines between the squares. For example, the first move might be to break the bar into a $3\times 5$ and a $5\times 5$ piece. If you can't break a piece, you lose.

Here's the invariant to consider: after each move, there is one more piece, and the game ends when there are 40. Thus, no matter what the moves are, the game is over in exactly 39 moves, so it is not really a game at all.

9. **Factor Into Primes.** Many problems about divisibility can be solved by recalling that every integer has a unique factorization into prime numbers. Show that between any pair of twin primes except 3 and 5, the number between them is a multiple of 6. (Twin primes are two prime numbers that differ by 2.) Any set of three successive numbers includes one that is a multiple of three. Since, (except in the case of 3 and 5) neither prime can be a multiple of three, the number between them must be. Every pair of twin primes consists of two odd numbers so the number between is a multiple of 2. Any number that is a multiple of both 2 and 3 is a multiple of 6.

# 5 Sample Problems

Here is a list of sample problems shamelessly copied from various contests. These problems are not for solution; instead, for each one think of as many approaches as you can that might work to solve it, and think of pictures or diagrams you might draw.

1. The year $1989 = 9 \cdot 13 \cdot 17$. Compute the next greater year that can be written as the product of three positive integers in arithmetic progression, given that the sum of those integers is 57.

2. Compute the value of:
$$\frac{(1990)^3 - (1000)^3 - (990)^3}{(1990)(1000)(990)}.$$

3. If $a + b = c$, $b + c = d$, $c + d = a$, and b is a positive integer, compute the greatest possible value for $a + b + c + d$.

4. A chord of constant length slides around in a semicircle. The midpoint of the chord and the projections of its ends upon the base form the vertices of a triangle. Prove that the triangle is isosceles and all possible such triangles are similar.

5. In how many ways can 10 be expressed as a sum of 5 nonnegative integers when order is taken into account? In other words, $0 + 3 + 2 + 0 + 5$ is different from $0 + 0 + 3 + 2 + 5$.

6. There are 100 soldiers in a detachment, and every evening three of them are on duty. Can it happen that after a certain period of time each soldier has shared duty with every other soldier exactly once?

7. The prime numbers $p$ and $q$ and the natural number $n$ satisfy the following equation:
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{pq} = \frac{1}{n}.$$
Find the numbers.

8. There are 7 glasses on a table—all standing upside down. One move consists of turning over any 4 of them. Is it possible to reach a situation where all the glasses are right side up?

9. Prove that if two quadrilaterals have the same midpoints for all of their sides, then their areas are equal.

10. For what values of $a$ does the system of equations:

$$\begin{aligned} x^2 &= y^2, \\ (x-a)^2 + y^2 &= 1 \end{aligned}$$

have exactly zero, one, two, three, and four solutions, respectively?

11. Show that:
$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

12. Using a straightedge and compass, construct a trapeziod given the lengths of all of its sides.

13. On every square of a $1997 \times 1997$ board is written either 1 or $-1$. For each row $i$, let $R_i$ be the product of the numbers in that row. Similarly, let $C_i$ be the product of the numbers in column $i$. Show that $\sum_{i=1}^{1997}(R_i + C_i)$ is never equal to zero.

14. The sequence $\{a_n\}_{n\geq 0}$ is defined as follows: $a_0$ is a positive rational number smaller than $\sqrt{1998}$, and if $a_n = p_n/q_n$ for some relatively prime integers $p_n$ and $q_n$, then
$$a_{n+1} = \frac{p_n^2 + 5}{p_n q_n}.$$
Show that $a_n < \sqrt{1998}$ for all $n$.

15. Mr. and Mrs. Adams recently attended a party at which there were three other couples. Various handshakes took place. No one shook hands with his/her own spouse, no one shook hands with the same person twice, and of course, no one shook his/her own hand.

    After all the handshaking was finished, Mr. Adams asked each person, including his wife, how many hands he or she had shaken. To his surprise, each gave a different answer. How many hands did Mrs. Adams shake?

# 6 Bibliography

Here is a short list of books on mathematical problem solving strategies.


1. Arthur Engel. *Problem-Solving Strategies.* Springer, New York, 1997.
2. Dmitri Fomin, Sergey Genkin, Ilia Itenberg. *Mathematical Circles (Russian Experience).* American Mathematical Society, Providence, 1996.
3. Loren C. Larson. *Problem-Solving Through Problems.* Springer-Verlag, New York, 1992.
4. George Pólya. *How to Solve It.* Doubleday, second edition, 1957.
5. Paul Zeitz. *The Art and Craft of Problem Solving.* John Wiley & Sons, Inc., New York, 1999.