Counting Poker Hands

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Problem: Find the number of poker hands of each type. For the purposes of this problem, a poker hand consists of 5 cards chosen from a standard pack of 52 (no jokers). Also for the purposes of this problem, the ace can only be a high card. In other words, the card sequence $A\clubsuit$, $2\diamondsuit$, $3\diamondsuit$, $4\spadesuit$, $5\heartsuit$ is not a straight, since the ace is a high card only.

Here are the definitions of the hands:

Royal flush: 10 through A in the same suit.

Example: $10\spadesuit$, $J\spadesuit$, $Q\spadesuit$, $K\spadesuit$, $A\spadesuit$.

Straight flush: 5 cards in sequence in the same suit, but not a Royal flush.

Example: $4\diamondsuit$, $5\diamondsuit$, $6\diamondsuit$, $7\diamondsuit$, $8\diamondsuit$.

Four of a kind: Four cards of the same rank.

Example: $Q\spadesuit$, $Q\heartsuit$, $Q\diamondsuit$, $Q\clubsuit$, $7\spadesuit$.

Full house: Three cards of one rank and two of another.

Example: $3\spadesuit$, $3\diamondsuit$, $3\clubsuit$, $9\heartsuit$, $9\diamondsuit$.

Flush: Five cards in the same suit that are not in sequence.

Example: $3\clubsuit$, $4\clubsuit$, $5\clubsuit$, $6\clubsuit$, $8\clubsuit$.

Straight: Five cards in sequence that are not all in the same suit.

Example: $6\clubsuit$, $7\clubsuit$, $8\diamondsuit$, $9\spadesuit$, $10\clubsuit$.

Three of a kind: Three cards of the same rank; the others of different rank.

Example: $J\clubsuit$, $J\diamondsuit$, $J\spadesuit$, $7\diamondsuit$, $K\heartsuit$.

Two pairs: Two pairs of cards.

Example: $5\heartsuit$, $5\spadesuit$, $8\diamondsuit$, $8\clubsuit$, $A\spadesuit$.

Pair: A single pair of cards.

Example: $3\clubsuit$, $3\diamondsuit$, $5\spadesuit$, $9\diamondsuit$, $Q\diamondsuit$.

Bust: A hand with none of the above.

Example: $2\clubsuit$, $4\spadesuit$, $6\diamondsuit$, $8\heartsuit$, $10\clubsuit$.

Total Hands: There are $\binom{52}{5} = 2598960$ total hands.

Royal Flush: There are 4 of these, since the hand is completely determined by the suit, and there are 4 possible suits.

Straight Flush: The hand is completely determined by the lowest card, which can be anything between 2 and 9 in any suit (if the lowest card is 10, it's a royal flush). There are 8 choices in 4 suits, making 32 straight flushes.

Four of a kind: There are 13 different ranks to choose for the four of a kind, and once that's picked, there are 48 additional cards to complete the hand. Therefore there are $13 \cdot 48 = 624$ hands.

Full house: There are 13 ways to choose the rank that appears 3 times, and then 12 ways to pick the pair. There are $\binom{4}{3}$ ways to pick the triplet, and $\binom{4}{2}$ ways to pick the pair. Thus there are $13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2} = 3744$ ways to make a full house.

Flush: There are 4 suits. Once the suit is chosen, there are $\binom{13}{5}$ ways to pick the cards, but 9 of those choices are either straight or royal flushes. Therefore, there are $4 \cdot (\binom{13}{5} - 9) = 5112$ flushes.

Straight: The lowest card in a straight can be of any rank between 2 and 10, so there are 9 possible lowest ranks. We then need to pick a card of each rank, which can be done in 4^5 ways. But 4 of those ways will all be in the same suit, so there are $9 \cdot (4^5 - 4) = 9180$ different straights.

Three of a kind: There are 13 kinds, from which you can pick the 3 cards in $\binom{4}{3}$ ways. Once they're picked, you need to choose 2 other cards. The first can be chosen in 48 ways, and once it's picked, the second in 44 ways. But those last two cards can be picked in either order, so we must divide by 2, giving a grand total of $13 \cdot \binom{4}{3} \cdot 48 \cdot 44/2 = 54912$.

Two pairs: The two ranks can be chosen in $\binom{13}{2}$ ways. Once they're picked, there are $\binom{4}{2}$ ways to pick the two particular cards of each rank. Finally, the final card can be picked in any of 44 ways, making the grand total: $\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44 = 123552$.

Pair: The rank can be chosen in 13 ways, and the two cards of that rank in $\binom{4}{2}$ ways. Next, we must choose 3 cards of different ranks, which can be done in 48, 44, and 40 ways. Since they can be picked in any order, we must divide by the number of permutations of the 3 cards, or $3!$. The total number of pairs is thus: $13 \cdot \binom{4}{2} \cdot 48 \cdot 44 \cdot 40/3! = 1098240$.

Bust: The easiest way to get this is to subtract all the above numbers from the grand total, but if we can calculate it in a different way, we have a check on all our numbers. The 5 cards must be chosen in 5 different ranks, so there are $52 \cdot 48 \cdot 44 \cdot 40 \cdot 36$ ways, but they can be picked in any order so we must divide by $5!$. This set includes the straights, flushes, straight flushes, and royal flushes, so subtract them out, for a grand total of $(52 \cdot 48 \cdot 44 \cdot 40 \cdot 36/5!) - 9180 - 5112 - 32 - 4 = 1303560$.

Check answers: $4 + 32 + 624 + 5112 + 9180 + 54912 + 123552 + 1098240 + 1303560 = 2598960$.