The 1 2 3 4 Problem

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Abstract

The first paragraph in the main part of this article poses a problem that can be approached by students who only need to be able to add, subtract, and multiply relatively small whole numbers. The problem is interesting because it suggests a large number of related problems of an almost unlimited range of difficulty, so it is a perfect topic for a math circle or for a math teachers' circle.

This article is meant to provide useful ideas for a leader of such a circle. It often contains "solutions" to the questions it poses, so if you want to use it as a circle leader, don't provide the circle participants with a copy of this article, at least until the circle session is complete.

1 Using the Problem in a Math Circle

1.1 The Problem

Starting with the numbers 1, 2, 3, and 4, what numbers can be formed from them using each of the numbers at most once, and using only addition, subtraction, multiplication and parentheses? You don't need to use all the numbers, and the numbers in your expression can appear in any order.

Here are a few examples showing how to obtain 3, 14, and 22 from those four numbers:

$$\begin{aligned} 3 &= 3, \text{ or } 1+2, \text{ or } 4-1 \\ 14 &= 2\cdot(4+3) \text{ or } 4\cdot(1+3)-2 \\ 22 &= 2\cdot(4\cdot 3-1) \end{aligned}$$

I discovered this problem here:

`http://www.mathteacherscircle.org/assets/session-materials/TatonOrderofOperations.pdf`

And found a lot more information here:

`http://sigmaa.maa.org/mcst/documents/G.pdf`

1.2 Suggestions for Leading the Circle

As a warm up, ask the students to see how many different numbers they can generate, using the rules above. Let them work alone, or in teams of two or three, as they choose. After ten minutes (or so: as long as there's a lot of active work going on, it's not a bad idea to let it continue) make a list combining results for everyone in the circle.

As you are collecting data, you'll probably want to list the expressions used to generate the various numbers. There are many ways to generate some numbers so leave room to list more than one way. Ask the participants if things like "$3+4$" and "$4+3$" are "different". See if there's an interest in keeping track of zero and negative results.

With this initial list, look for gaps and perhaps allow some time for the groups to try to fill in the gaps. Obtain suggestions for questions that might be asked. Here are some possible questions, but obviously lots of others may come up.

1. What is the largest possible number?
2. What is the smallest possible number (the most negative)?
3. How many different results are possible? How do you know that you have found them all?
4. What is the smallest positive number that cannot be made? How do you know that it cannot be made?
5. Some results can be achieved in many ways. For example, how many ways can you make 3? 9? Other numbers?
6. What happens if we do not allow negative numbers to appear in the calculation? (In other words, if subtraction cannot be used to produce a negative number during the calculation.)
7. Which number(s) can be made in the most different ways?
8. What is a "different way"? (In other words, is $3 + 5$ different from $5 + 3$? Perhaps we need some way to organize the structure of the expression so that calculations that are basically the same can be identified.)
9. What is the smallest number that can be made only in one way?
10. We can obviously modify the question to use larger (or smaller) sets, like $\{1, 2, 3, 4, 5\}$ or $\{1, 2, 3\}$.
11. Another way to complicate or simplify the problem is to remove or add operations. What if only addition is allowed? Only multiplication? Only addition and multiplication[1]? What if we include more operations, like division or exponentiation?
12. If we use sets other than those made of successive integers like $\{1, 2, 3, \ldots\}$, what are the "best" sets in terms of making the most numbers? The largest missing number? (See Section 4.)
13. How can we know we have a complete solution (to any of the problems above)?
14. For people with an interest in computer calculations, how can you write a program that will solve problems like the ones above? (Some of the suggestions in Section 2 can be fleshed out to form algorithms that will do the job, but to do a good job, the programmer will need to be moderately sophisticated.)

Here is list of useful strategies that can be used to answer some of the questions above. Try to solicit some of these ideas from the circle participants as well. (Some of these strategies are generally useful for all sorts of problem solving.)

1. Make an organized list of results
2. Work backwards
3. Consider factorization
4. Look at simpler problems
5. Use previous results

[1]This is a *very* fruitful direction to go. See Section 3.

For the presenter's information, here is a complete list of the 55 different numbers that can be obtained starting with the numbers $\{1, 2, 3, 4\}$. In Section 5 you will find expressions that generate all the positive numbers in this set.

$$\begin{aligned}
\{&-23, -22, -21, -20, -19, -18, -17, -16, -15, -14,\\
&-13, -12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2,\\
&-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\\
&17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 32, 36\}
\end{aligned}$$

The answer to one of the earlier questions is that 29 is the smallest positive number that cannot be obtained from the set $\{1, 2, 3, 4\}$.

To be certain that the list above is complete is a difficult problem. If you want to consider it, it's a great idea to look first at simpler problems, starting with the simplest: Use the same rules but use only numbers in this set: $\{1\}$. Now solve it again using numbers in this set: $\{1, 2\}$. Finally, solve it with this set: $\{1, 2, 3\}$. These three problems are far more tractable.

Working out a similar list for $\{1, 2, 3, 4, 5\}$ is very difficult, at least without computer assistance. Here is a complete list of all 219 possibilities:

$$\begin{aligned}
\{&-119, -118, -117, -116, -115, -114, -112, -110, -108,\\
&-105, -100, -99, -96, -95, -90, -89, -87, -85, -84, -83,\\
&-81, -80, -78, -77, -75, -73, -72, -70, -69, -68, -67,\\
&-66, -65, -64, -63, -62, -61, -60, -59, -58, -57, -56,\\
&-55, -54, -53, -52, -51, -50, -49, -48, -47, -46, -45,\\
&-44, -43, -42, -41, -40, -39, -38, -37, -36, -35, -34,\\
&-33, -32, -31, -30, -29, -28, -27, -26, -25, -24, -23,\\
&-22, -21, -20, -19, -18, -17, -16, -15, -14, -13, -12,\\
&-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4,\\
&5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23,\\
&24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40,\\
&41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57,\\
&58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74,\\
&75, 77, 78, 80, 81, 82, 83, 84, 85, 87, 88, 89, 90, 91, 93, 95, 96,\\
&99, 100, 101, 104, 105, 108, 110, 112, 114, 115, 116, 117, 118,\\
&119, 120, 121, 122, 123, 124, 125, 126, 128, 130, 132, 135,\\
&140, 144, 150, 160, 180\}
\end{aligned}$$

In this list the smallest positive number that cannot be obtained is 76.

There's plenty of material in just this section to lead a circle for an hour or two. In the sections that follow, you will find more information related to some of the questions above, and occasionally related interesting mathematical directions you can explore.

2 How To Make A Complete List

How can we be sure that we have all the possibilities for the original problem? Just because you have looked and looked, maybe there is some expression that you forgot about.

Perhaps the most straightforward approach would be to make a complete list of all possible expressions that can be formed. Begin with variables rather than the numbers themselves since they are easier to list, and when you have the complete list, substitute the numerical values into those expressions in every possible order.

Since a useful problem solving strategy is often to approach a hard problem by looking at simpler versions of it, start with an easier problem: How many expressions can be formed with one variable? With two? With three? For the set $\{1, 2, 3\}$ (or any set with three numbers), here's the list of all possible expressions (including those with fewer than three variables), where each V stands for one of the three numbers and when more than one V appears, each represents a different number:

$$
\begin{array}{l}
V \\
(V+V),\ (V-V),\ (V\cdot V) \\
(V+V+V),\ (V+V-V),\ (V-V-V),\ (V+V\cdot V),\ (V-V\cdot V), \\
(V\cdot V-V),\ (V\cdot V\cdot V),\ (V\cdot(V+V)),\ (V\cdot(V-V))
\end{array}
$$

Now we can find all possible values by plugging in every list of length 1 into the first expression (there are three of these), every list of length 2 into the second expressions (there are six of these lists since $\{1, 2\}$ and $\{2, 1\}$ are different), and every list of length 3 (six again: all of the permutations of a set of three numbers) into the expressions with three variables. This will require $3 + 3 \cdot 6 + 9 \cdot 6 = 75$ evaluations.

How are we even sure that we have all the expressions? There is only one of length 1, so that's easy. To find those of length 2, we need to construct all possible sums, differences, and products of terms of length 1. For expressions with three variables, we need to take all sums, products, and differences of expressions of length 1 with expressions of length 2, and in addition, differences of expressions of length 2 with those of length 1.

This will yield 12 expressions which is more than the 9 listed above. Why? Because some are equivalent, in the sense that since we're going to substitute every permutation of the three elements into every expression, expressions like $V - V + V$ and $V + V - V$ are equivalent since when we substitute $1, 2, 3$ (in that order) into the first, we obtain $1-2+3$ and when we substitute $1, 3, 2$ or $3, 1, 2$ into the second, we obtain the equivalent $1 + 3 - 2$ and $3 + 1 - 2$.

To find all expressions with four variables, we need to add, subtract, and multiply all expressions with one variable against all those with three variables. Then we have to do the same thing with all pairs of expressions containing exactly two variables. As the number of variables increases, the number of possible expressions increases exponentially. The problem of duplicate equivalent expressions gets *much* worse as the number of variables increases, and tremendous amounts of additional calculations need to be done to find the true results.

The big problem we encounter is that addition and multiplication are commutation, but subtraction is not. When we eliminate that problem, things become a lot easier. See Section 3.

Another method would be to start with the smallest subsets and work our way up. Let's find all the possibilities for the set $\{1, 2, 3\}$. We begin with a list of all possible results for the singletons:

$$
\begin{array}{ccc}
\{1\} & \longrightarrow & \{1\} \\
\{2\} & \longrightarrow & \{2\} \\
\{3\} & \longrightarrow & \{3\}
\end{array}
$$

(Interpret this to mean that the list of all possible results from the set that just contains the number 1 contains one element; namely, 1.)

This doesn't say much. It basically says that if you have a set with a single number in it, the only result you can obtain by adding, subtracting or multiplying elements is that single number.

The next stage is more interesting. For example, to see the list of numbers we can obtain for $\{1, 3\}$ we take all possible elements from the results available from $\{1\}$ and from $\{2\}$ and add, subtract, and multiply them, or just include the numbers already generated. This yields:

$$\begin{array}{lcl} \{1,2\} & \longrightarrow & \{-1,1,2,3\} \\ \{1,3\} & \longrightarrow & \{-2,1,3,4\} \\ \{2,3\} & \longrightarrow & \{-1,1,2,3,5,6\} \end{array}$$

(As before, the top line means that if we make all possible expressions from the set containing 1 and 2, there are four possible results: -1, 1, 2, and 3.)

The final step takes longer. Results from the full set $\{1,2,3\}$ can be obtained by combining the results from any two subsets:

$$\begin{array}{lcl} \{1,2,3\} = \{1\} \cup \{2,3\} & \longrightarrow & \{-5,-4,-2,-1,0,1,2,3,4,5,6,7\} \\ \{1,2,3\} = \{2\} \cup \{1,3\} & \longrightarrow & \{-4,-2,0,1,2,3,4,5,6,8\} \\ \{1,2,3\} = \{3\} \cup \{1,2\} & \longrightarrow & \{-4,-3,-1,0,1,2,3,4,6,9\} \end{array}$$

The union of all the results is, therefore:

$$\{1,2,3\} \quad \longrightarrow \quad \{-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,8,9\}$$

This set contains 15 different elements, and the smallest number that cannot be obtained is 10.

This process gets painful as the size of the set goes up since a set of size n has $n!$ subsets, so for $n = 4$ we would have to combine the results from 12 different pairs of subsets. For $n = 4$ we would finally obtain this list of possible values which was listed in the first section:

$$\begin{array}{l} \{-23,-22,-21,-20,-19,-18,-17,-16,-15,-14, \\ \;\; -13,-12,-11,-10,-9,-8,-7,-6,-5,-4,-3,-2, \\ \;\; -1,0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16 \\ \;\; 17,18,19,20,21,22,23,24,25,26,27,28,30,32,36\} \end{array}$$

For the same problem for the set $\{1,2,3,4\}$ the calculations are quite a bit more involved. Beginning with the trivial subsets $\{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$, we need to work out the achievable numbers for all six pairs: $\{1,2\}$, $\ldots \{3,4\}$, then the four triples, and finally for the full set. Here is a list of the pairs that would need to be joined for the full set:

$$\begin{array}{rcl} \{1,2,3,4\} & = & \{1\} \cup \{2,3,4\} = \{2\} \cup \{1,3,4\} = \{3\} \cup \{1,2,4\} = \{4\} \cup \{1,2,3\} \\ & = & \{1,2\} \cup \{3,4\} = \{1,3\} \cup \{2,4\} = \{1,4\} \cup \{2,3\}. \end{array}$$

3 Addition and Multiplication Only

If we simplify the problem to allow only additions and multiplications, the problem becomes quite a bit easier:

1. We don't have to worry about negative numbers.
2. Both operations are commutative, so there are many fewer basic forms.

Try to find all the values that can be achieved using addition and multiplication only for the following sets of numbers: $\{1\}$, $\{1,2\}$, $\{1,2,3\}$, $\{1,2,3,4\}$.

It is pretty easy to make complete lists of all the possible forms that expressions can take for the first few numbers of variables. If we denote by V a variable, then here is a list of all possible forms for the numbers of variables ranging from 1 to 4:

$$
\begin{array}{rcl}
V & \longleftrightarrow & V \\
V \cdot V & \longleftrightarrow & V + V \\
V \cdot V \cdot V & \longleftrightarrow & V + V + V \\
V \cdot (V + V) & \longleftrightarrow & V + (V \cdot V) \\
V \cdot V \cdot V \cdot V & \longleftrightarrow & V + V + V + V \\
V \cdot V \cdot (V + V) & \longleftrightarrow & V + V + (V \cdot V) \\
V \cdot (V + V + V) & \longleftrightarrow & V + (V \cdot V \cdot V) \\
V \cdot (V + (V \cdot V)) & \longleftrightarrow & V + (V \cdot (V + V)) \\
(V + V) \cdot (V + V) & \longleftrightarrow & (V \cdot V) + (V \cdot V)
\end{array}
$$

There are a few things to note about the list above:

1. The first line duplicates the simplest term; namely, V.
2. There are a few extra parentheses in the expressions that would not normally appear in a mathematical formula because we know that multiplication takes precedence over addition. Without such a rule, all the parentheses in the table would be necessary. Also, we usually omit the "$\cdot$" indication of multiplication so we would typically write something like VVV (or V^3) instead of $V \cdot V \cdot V$. In the expressions above, every operation is explicitly indicated.
3. There is an exact correspondence between the terms on the left and the terms on the right: we simply replace the "$+$" operator with the "$\cdot$" operator and vice-versa. A mathematician would call the two versions "duals."
4. Notice that all the patterns on the left have the form of a pure product of some number of items that are either simple terms or sums of terms. Those on the right are the opposite: sums of simple terms or products of terms. Another way to think of this is that when you evaluate an expression on the left, the final operation has to be a multiplication; on the right, it has to be an addition.

It's a good exercise to generate all the possible expressions with 5 variables and to pair them up, as above. You should be able to find 12 pairs, or 24 possible expressions.

3.1 Series-Parallel Networks

After having done this, we have an interesting sequence of numbers; namely, the number of expressions using only "$+$" and "$\cdot$". There are $1, 2, 4, 10$, and 24 expressions with $1, 2, 3, 4$, and 5 variables, respectively. When you have a sequence of numbers like this it is often very interesting to plug it into the "Online Encyclopedia of Integer Sequences" at:

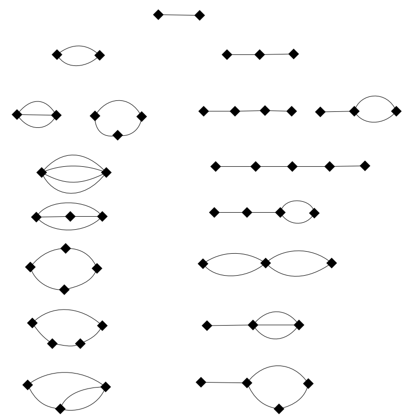
`http://oeis.org/`

In this case we find:

`http://oeis.org/A000084`

matches our sequence, and it's entitled, "Number of series-parallel networks with n unlabeled edges." Also included are references for formulas to generate the series as well as pointers to documents that describe exactly what is meant by the title.

Here is an illustration of all the series-parallel networks of length 1 to 4. Think of the lines connecting the points as the items we are counting:



The networks on the left are basically parallel and the ones on the right (their duals) are basically series. In other words, the paths on the left consist of some number of simpler (series) paths in parallel and the ones on the right consist of some number of (parallel) paths in series. Of course we consider the simplest path to be both parallel and series so it can be a component of either type. Also note that the order of placement of the paths doesn't matter in the sense that the series networks consisting of four parallel networks of types A, A, A, and B could look like $AAAB$, $AABA$, $ABAA$ or $BAAA$. (They combine in exactly the same as commutative elements in mathematics.)

If we think of paths in parallel as corresponding to multiplication and those in series to addition[2] then the five networks that consist of four lines (starting on the fourth row from the top) correspond to the following expressions:

$$\begin{array}{rcl}
V\cdot V\cdot V\cdot V & \longleftrightarrow & V+V+V+V \\
V\cdot V\cdot (V+V) & \longleftrightarrow & V+V+(V\cdot V) \\
(V+V)\cdot (V+V) & \longleftrightarrow & (V\cdot V)+(V\cdot V) \\
V\cdot (V+V+V) & \longleftrightarrow & V+(V\cdot V\cdot V) \\
V\cdot (V+(V\cdot V)) & \longleftrightarrow & V+(V\cdot (V+V))
\end{array}$$

If we would like to generate the next set of networks (with 5 lines), we could do it as follows: We can generate all the next series networks by combining all possible sets of parallel networks that have a total of 5 lines. Then construct the duals of each of those. We'll do that in the following section, but using the expressions themselves.

The `oeis.org` web page provides a formula for computing the number of expressions there are, and the first few of those numbers are, 1, 2, 4, 10, 24, 66, 180, 522, 1532, 4624, 14136, 43930, 137908, 437502, $\ldots$. This is a relatively small number and a complete evaluation of the possible results would take the following number of evaluations: $1\cdot 1!$, $2\cdot 2!$, $4\cdot 3!$, $10\cdot 4!$, $24\cdot 5!$, $\ldots$.

[2] We could make the opposite assumption and come to exactly the same result.

3.2 Generating All Possible Expressions

Suppose we know all the ways to generate expressions with 4 variables (which we have above) and we want to generate all the ways to generate expressions with 5 variables. One way to do this would be to generate all of the possible versions of sums of products (sums of versions on the left of the table above) and then take their duals to find the product versions.

The number 5 can be represented as a sum of numbers in the following ways:

$$\begin{aligned}
5 &= 1+1+1+1+1\\
5 &= 1+1+1+2\\
5 &= 1+2+2\\
5 &= 1+1+3\\
5 &= 2+3\\
5 &= 1+4\\
5 &= 5
\end{aligned}$$

The 7 expressions above are called the "partitions" of 5. We can interpret this list as follows to produce sum expressions from shorter product expresions:

1. We can make one by adding 5 product expressions of length 1.
2. We can make one by adding 3 product expressions of length 1 and one of length 2.
3. We can make one by adding 1 product expression of length 1 and two of length 2.
4. et cetera...

We don't need to use the last partition ($5 = 5$) since we're building new expressions up from smaller ones. For product-only expressions of lengths 1 and 2, there is only 1 type of each. There are 2 of length 3 and 5 of length 4. Thus in total, there are:

$$1+1+1+2+2+5=12$$

ways to make sums of pure products of terms that have 5 total variables. Since all of these can be converted to expressions as pure products by duality, there are 24 expressions of length 5.

Here they are, together with their duals, this time with the sum versions on the left listed in the order described in the paragraph above:

$$\begin{aligned}
V+V+V+V+V &\longleftrightarrow V\cdot V\cdot V\cdot V\cdot V\\
V+V+V+(V\cdot V) &\longleftrightarrow V\cdot V\cdot V\cdot(V+V)\\
V+(V\cdot V)+(V\cdot V) &\longleftrightarrow V\cdot(V+V)\cdot(V+V)\\
V+V+(V\cdot V\cdot V) &\longleftrightarrow V\cdot V\cdot(V+V+V)\\
V+V+V\cdot(V+V) &\longleftrightarrow V\cdot V\cdot V+(V\cdot V)\\
(V\cdot V)+(V\cdot V\cdot V) &\longleftrightarrow (V+V)\cdot(V+V+V)\\
(V\cdot V)+V\cdot(V+V) &\longleftrightarrow (V+V)\cdot V+(V\cdot V)\\
V+(V\cdot V\cdot V\cdot V) &\longleftrightarrow V\cdot(V+V+V+V)\\
V+(V\cdot V\cdot(V+V)) &\longleftrightarrow V\cdot(V+V+(V\cdot V))\\
V+(V\cdot(V+V+V)) &\longleftrightarrow V\cdot(V+(V\cdot V\cdot V))\\
V+(V\cdot(V+(V\cdot V))) &\longleftrightarrow V\cdot(V+(V\cdot(V+V)))\\
V+((V+V)\cdot(V+V)) &\longleftrightarrow V\cdot((V\cdot V)+(V\cdot V))
\end{aligned}$$

Of course starting with this list we can generate all expressions with 6 variables if we first make a list of the partitions of 6 and proceed as before, adding together all the combinations of product terms that have 6 total variables and then constructing their duals to obtain the complete list. There are 33 pairs, or 66 total expressions, and the process can be continued.

3.3 Counting by Computer: Stacks and Polish Notation

We could enumerate the possible expression results by generating formulas as in the list above and then evaluating each with all possible assignments of numbers to the V's, but there is a much more convenient form to express the expressions than the one above that has nested parentheses. A computer evaluation of the expressions above would involve first finding the deepest parenthesis nesting, evaluating the contained expression, and working its way out.

A much easier way to describe expressions is with a sequence of stack operations.

A "stack" is a list of numbers that is only accessible from one end. We can add numbers to the top of the stack, operate on numbers on top of the stack, or remove numbers from the top of the stack. It's sort of like a stack of trays in a cafeteria: it's easy to access the top trays: to add new ones to the top or to take them off the top, but it is very difficult to access the trays on the bottom.

When we add a number to the top of our stack, we say we are pushing a number onto the stack. When we remove one, we say we are popping the stack. Finally, we can perform operations on the top of the stack; in this case, addition and multiplication. The "add" operation pops the top two numbers off the top, adds them, and pushes the result back onto the stack. The "multiply" operation is the same, except that the two popped numbers are multiplied and the result pushed back onto the stack[3].

We can describe a computation as a simple list of numbers and operations. To perform the computation, we just take items from the list in order, and if the item is a number, we push it on the stack. If it is an operation, we perform that operation on the stack. When we reach the end of the list, the number remaining on the top of the stack is the result of the computation.

For example, to evaluate the computation described by the list: "$2\ 3 + 4\ \cdot$" we first push a 2, then a 3, so at that point there are two numbers on the stack: 3 on top and 2 below it. The $+$ takes the top two numbers, 2 and 3, adds them to obtain 5, and pushes the 5 back on the stack. Next a 4 is pushed and the final $\cdot$ multiplies the 4 and 5 yielding 20 and pushes the 20 back on the stack. The result of the calculation is the top of the stack; namely, 20. This list effectively calculates the expression $(2 + 3) \cdot 4$.

The beauty of a stack representation of expression is that parentheses are never needed. It is a great exercise to practice the generation of a stack expression from a standard expression that contains nested parentheses.

Of course there are invalid lists, like "$1\ +$" where the add will fail since there is only one item on the stack. Also, there may be more than one way to express a calculation. The following two lists generate the same $1 + 2 + 3$:

$$1\,2\ +\ 3\,+$$

and

$$1\,2\,3\ +\ +$$

The linear notation of mixed numbers and operations to represent a calculation is called "Polish notation," or sometimes "reverse Polish notation" because you name the operator after you name the operands. One very nice feature of this notation is that if you wish to combine two expressions by multiplying them or adding them, you simply concatenate the two lists and append a "$+$" or a "$\cdot$".

For example, the list "4, 3, $+$" represents the calculation to obtain $4+3$ and "5, 6, $+$" represents the calculation of $5 + 6$. To represent the calculation of:

$$(4 + 3) \cdot (5 + 6)$$

[3] Of course we can also construct stack expressions using subtraction as well, but we'll stick to addition and multiplication here.

we just concatenate the two lists and add a "$\cdot$":

$$4\,3\ +\ 5\,6\ +\ \cdot$$

If we use "V" to represent a number, then, for example, the expression:

$$VV \cdot V + V \cdot V +$$

will represent every calculation of the form $(V \cdot V + V) \cdot V + V$, and every possible substitutions of different numbers from the original list would represent all the possible calculations from an expression of this form.

(Note that a list like "V" represents a "calculation"; namely, output the number. Technically, put the number on the stack, but now you're at the end of the list, so the number on the top of the stack is your result.)

Here is a list of all possible calculations with up to four numbers together with the corresponding Polish notation versions:

$$\begin{array}{rcl}
V & \longleftrightarrow & V \\
V \cdot V & \longleftrightarrow & VV\,\cdot \\
V + V & \longleftrightarrow & VV\,+ \\
V \cdot V \cdot V & \longleftrightarrow & VV \cdot V\,\cdot \\
V + V + V & \longleftrightarrow & VV + V\,+ \\
V \cdot (V + V) & \longleftrightarrow & VVV + \cdot \\
V + (V \cdot V) & \longleftrightarrow & VVV \cdot + \\
V \cdot V \cdot V \cdot V & \longleftrightarrow & VV \cdot V \cdot V\,\cdot \\
V + V + V + V & \longleftrightarrow & VV + V + V\,+ \\
V \cdot V \cdot (V + V) & \longleftrightarrow & VV \cdot VV + \cdot \\
V + V + (V \cdot V) & \longleftrightarrow & VV + VV \cdot + \\
V \cdot (V + V + V) & \longleftrightarrow & VVV + V + \cdot \\
V + (V \cdot V \cdot V) & \longleftrightarrow & VVV \cdot V \cdot + \\
V \cdot (V + (V \cdot V)) & \longleftrightarrow & VVVV \cdot + \cdot \\
V + (V \cdot (V + V)) & \longleftrightarrow & VVVV + \cdot + \\
(V + V) \cdot (V + V) & \longleftrightarrow & VV + VV + \cdot \\
(V \cdot V) + (V \cdot V) & \longleftrightarrow & VV \cdot VV \cdot +
\end{array}$$

After the first line, every pair of calculations are dual, and note how easy it is to obtain the dual of a calculation in Polish notation: leave all the V's exactly as they are, and swap the "$\cdot$"s with the "$+$"s.

This method of representing expressions makes computer calculation much easier.

Another way to do it, of course, is to begin with the smallest subsets of the original set of numbers and recursively generate the lists of possible results from combinations of all possible subsets of those sets. Most of the numbers in the next session were computed using programs of both of the sorts above.

4 Better Initial Sets

Up to now we have been generating numbers starting with the initial positive integers: $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, and so on. Can we do better with different initial sets?

The answer is yes, we can do quite a bit better. It is a reasonable problem to look for the best set of three numbers by hand, but searching for the best sets of four or more numbers probably requires some computer work.

Here are the answers for smaller set sizes:

The set $\{1\}$ generates only the number 1.

The set $\{1, 3\}$ generates all the positive whole numbers up to and including 4.

The set $\{2, 3, 10\}$ generates all the positive whole numbers up to and including 17.

The set $\{2, 3, 4, 27\}$ or the set $\{2, 3, 10, 41\}$ generates all the positive whole numbers up to and including 69.

The set $\{2, 3, 4, 84, 111\}$ generates all the positive whole numbers up to and including 404.

Although it is not easy to show that the last two are correct, the circle members can try to verify at least a few of the numbers. Note that the solution for a set of five is similar to one of the optimal solutions for four, since every solution for a set of four can be converted to a solution for a set of five by replacing the 27 with $111 - 84$.

See:

`http://oeis.org/A141494`

Obviously, the same question can be asked about the simpler situation when only multiplication and addition are allowed. In this case, we obtain the following results:

The set $\{1\}$ generates only the number 1.

The set $\{1, 2\}$ generates all the positive whole numbers up to and including 3.

The set $\{1, 2, 4\}$ generates all the positive whole numbers up to and including 10.

The set $\{1, 2, 4, 5\}$ generates all the positive whole numbers up to and including 32.

The set $\{1, 2, 3, 8, 26\}$ generates all the positive whole numbers up to and including 98.

The set $\{1, 2, 3, 7, 9, 41\}$ generates all the positive whole numbers up to and including 462.

5 Solutions

One good exercise for students is to try to find expressions that result in target numbers. The "solutions" in this section show at least one way to obtain all the numbers from 1 up until the first number that can not be achieved. Searching for such expressions, especially with the ones toward the end with larger numbers, will provide a lot of arithmetic drill. Perhaps the class/circle can try to find, over the course of a few days, examples that generate all the results on these pages.

Here is a table of the first number that cannot be formed and the total number of results that can be formed from the following sets of numbers using multiplication, addition, and subtraction:

$\{1\}$	2	1
$\{1, 2\}$	4	4
$\{1, 2, 3\}$	10	15
$\{1, 2, 3, 4\}$	29	55
$\{1, 2, 3, 4, 5\}$	76	219
$\{1, 2, 3, 4, 5, 6\}$	284	1003
$\{1, 2, 3, 4, 5, 6, 7\}$	1413	5395
$\{1, 2, 3, 4, 5, 6, 7, 8\}$	7187	31140
$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$	38103	205899

Here are some solutions for the sets above up to length 5.

1	1	26	$2 \cdot (1 + 3 \cdot 4)$	51	$(1 + 2) \cdot (4 \cdot 5 - 3)$
2	2	27	$(1 + 2 \cdot 4) \cdot 3$	52	$(3 \cdot 5 - 2) \cdot 4$
3	$1 + 2$	28	$(2 \cdot 3 + 1) \cdot 4$	53	$(3 \cdot 5 - 2) \cdot 4 + 1$
4	$1 + 3$	29	$(1 + 2) \cdot 3 + 4 \cdot 5$	54	$(2 \cdot 4 + 3) \cdot 5 - 1$
5	$2 + 3$	30	$2 \cdot 3 \cdot (1 + 4)$	55	$(3 \cdot 4 - 1) \cdot 5$
6	$2 \cdot 3$	31	$2 \cdot 3 \cdot 5 + 1$	56	$1 + (2 \cdot 4 + 3) \cdot 5$
7	$2 \cdot 3 + 1$	32	$2 \cdot 4 \cdot (1 + 3)$	57	$5 \cdot 4 \cdot 3 - 2 - 1$
8	$(1 + 3) \cdot 2$	33	$(3 + 5) \cdot 4 + 1$	58	$5 \cdot 4 \cdot 3 - 2$
9	$(1 + 2) \cdot 3$	34	$(3 + 5) \cdot 4 + 2$	59	$5 \cdot 4 \cdot 3 - 1$
10	$(1 + 4) \cdot 2$	35	$(3 + 5) \cdot 4 + 1 + 2$	60	$5 \cdot 4 \cdot 3$
11	$3 \cdot 4 - 1$	36	$3 \cdot 4 \cdot (1 + 2)$	61	$5 \cdot 4 \cdot 3 + 1$
12	$3 \cdot 4$	37	$2 \cdot 4 \cdot 5 - 3$	62	$5 \cdot 4 \cdot 3 + 2$
13	$3 \cdot 4 + 1$	38	$2(4 \cdot 5 - 1)$	63	$5 \cdot 4 \cdot 3 + 2 + 1$
14	$2 \cdot (3 + 4)$	39	$2 \cdot 4 \cdot 5 - 1$	64	$2 \cdot 4 \cdot (5 + 3)$
15	$(1 + 4) \cdot 3$	40	$2 \cdot 4 \cdot 5$	65	$2 \cdot 4 \cdot (5 + 3) + 1$
16	$(1 + 3) \cdot 4$	41	$2 \cdot 4 \cdot 5 + 1$	66	$(5 + 1) \cdot (4 \cdot 2 + 3)$
17	$(2 + 4) \cdot 3 - 1$	42	$(3 + 4 - 1) \cdot (2 + 5)$	67	$4 \cdot (5 \cdot 3 + 2) - 1$
18	$(2 + 4) \cdot 3$	43	$2 \cdot 4 \cdot 5 + 3$	68	$4 \cdot (5 \cdot 3 + 2)$
19	$(2 + 4) \cdot 3 + 1$	44	$(1 + 2 + 3 + 5) \cdot 4$	69	$2 \cdot 5 \cdot (4 + 3) - 1$
20	$(2 + 3) \cdot 4$	45	$(2 + 3) \cdot (4 + 5)$	70	$2 \cdot 5 \cdot (4 + 3)$
21	$(2 + 3) \cdot 4 + 1$	46	$2 \cdot (4 \cdot 5 + 3)$	71	$2 \cdot 5 \cdot (4 + 3) + 1$
22	$2(3 \cdot 4 - 1)$	47	$2 \cdot (4 \cdot 5 + 3) + 1$	72	$(4 \cdot 2 + 1) \cdot (5 + 3)$
23	$2 \cdot 3 \cdot 4 - 1$	48	$(2 + 4) \cdot (3 + 5)$	73	$3 \cdot 5 \cdot (1 + 4) - 2$
24	$2 \cdot 3 \cdot 4$	49	$(2 + 5) \cdot (3 + 4)$	74	$4 \cdot 3 \cdot (1 + 5) + 2$
25	$2 \cdot 3 \cdot 4 + 1$	50	$(1 + 4) \cdot 2 \cdot 5$	75	$5 \cdot 3 \cdot (1 + 4)$

Here is a table of the first number that cannot be formed and the total number of results that can be formed from the following sets of numbers using multiplication and addition only:

$\{1\}$	2	1
$\{1, 2\}$	4	3
$\{1, 2, 3\}$	10	9
$\{1, 2, 3, 4\}$	22	29
$\{1, 2, 3, 4, 5\}$	58	108
$\{1, 2, 3, 4, 5, 6\}$	233	465
$\{1, 2, 3, 4, 5, 6, 7\}$	827	2314
$\{1, 2, 3, 4, 5, 6, 7, 8\}$	3359	12605
$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$	16631	78580

Here are some solutions for the sets above up to length 5.

1	1	20	$(2+3)\cdot 4$	39	$(4\cdot 2+5)\cdot 3$
2	2	21	$(3+2)\cdot 4+1$	40	$5\cdot 4\cdot 2$
3	$2+1$	22	$5\cdot 4+2$	41	$5\cdot 4\cdot 2+1$
4	$3+1$	23	$5\cdot 4+3$	42	$(5\cdot 4+1)\cdot 2$
5	$3+2$	24	$4\cdot 3\cdot 2$	43	$5\cdot 4\cdot 2+3$
6	$3\cdot 2$	25	$4\cdot 3\cdot 2+1$	44	$(5\cdot 2+1)\cdot 4$
7	$3\cdot 2+1$	26	$(4\cdot 3+1)\cdot 2$	45	$(2+1)\cdot 5\cdot 3$
8	$(3+1)\cdot 2$	27	$(4\cdot 2+1)\cdot 3$	46	$(5\cdot 4+3)\cdot 2$
9	$(2+1)\cdot 3$	28	$(3\cdot 2+1)\cdot 4$	47	$(5\cdot 4+3)\cdot 2+1$
10	$2\cdot 3+4$	29	$4\cdot 3\cdot 2+5$	48	$(5+1)\cdot 4\cdot 2$
11	$2\cdot 4+3$	30	$(4+1)\cdot 3\cdot 2$	49	$(4+3)\cdot(5+2)$
12	$4\cdot 3$	31	$5\cdot 3\cdot 2+1$	50	$(4+1)\cdot 5\cdot 2$
13	$4\cdot 3+1$	32	$(3+1)\cdot 2\cdot 4$	51	$(5+1)\cdot 4\cdot 2+3$
14	$4\cdot 3+2$	33	$(5+3)\cdot 4+1$	52	$(5\cdot 2+3)\cdot 4$
15	$(4+1)\cdot 3$	34	$5\cdot 3\cdot 2+4$	53	$(4+1)\cdot 5\cdot 2+3$
16	$(3+1)\cdot 4$	35	$(4+3)\cdot 5$	54	$(5+4)\cdot 3\cdot 2$
17	$(4+1)\cdot 3+2$	36	$(2+1)\cdot 4\cdot 3$	55	$(4\cdot 2+3)\cdot 5$
18	$(2+4)\cdot 3$	37	$(4+3)\cdot 5+2$	56	$(4\cdot 2+3)\cdot 5+1$
19	$(4+2)\cdot 3+1$	38	$(5\cdot 3+4)\cdot 2$	57	$(5\cdot 3+4)\cdot(2+1)$

Here are some solutions using numbers in the set $\{2, 3, 4, 27\}$ using addition, subtraction, and multiplication for all results up to 69.

1	$3-2$	24	$27-3$	47	$2\cdot 27-3-4$
2	2	25	$27-2$	48	$2\cdot(27-3)$
3	3	26	$27-3+2$	49	$2\cdot(27-4)+3$
4	4	27	27	50	$2\cdot 27-4$
5	$2+3$	28	$27+3-2$	51	$2\cdot 27-3$
6	$2\cdot 3$	29	$27+2$	52	$2\cdot(27-4+3)$
7	$4+3$	30	$27+3$	53	$2\cdot 27+3-4$
8	$4\cdot 2$	31	$27+4$	54	$2\cdot 27$
9	$2+3+4$	32	$27+2+3$	55	$2\cdot 27+4-3$
10	$4\cdot 3-2$	33	$27+2\cdot 3$	56	$2\cdot(27+4-3)$
11	$2\cdot 4+3$	34	$27+3+4$	57	$2\cdot 27+3$
12	$3\cdot 4$	35	$27+2\cdot 4$	58	$2\cdot 27+4$
13	$27-(4\cdot 3+2)$	36	$27+2+3+4$	59	$2\cdot(27+4)-3$
14	$2\cdot(3+4)$	37	$27+4\cdot 3-2$	60	$2\cdot(27+3)$
15	$27-3\cdot 4$	38	$27+2\cdot 4+3$	61	$2\cdot 27+3+4$
16	$27-2\cdot 4-3$	39	$27+3\cdot 4$	62	$2\cdot(27+4)$
17	$27-3\cdot 4+2$	40	$2\cdot(27-3-4)$	63	$3\cdot(27-4-2)$
18	$(2+4)\cdot 3$	41	$27+2\cdot(3+4)$	64	$2\cdot(27+3)+4$
19	$27-2\cdot 4$	42	$2\cdot 27-3\cdot 4$	65	$2\cdot(27+4)+3$
20	$(2+3)\cdot 4$	43	$2\cdot(27-4)-3$	66	$2\cdot 27+3\cdot 4$
21	$27-2\cdot 3$	44	$2\cdot(27-3)-4$	67	$3\cdot(27-4)-2$
22	$27-2-3$	45	$27+3\cdot(2+4)$	68	$2\cdot(27+3+4)$
23	$27-4$	46	$2\cdot(27-4)$	69	$3\cdot(27-4)$

Here are solutions using numbers in the set $\{1, 2, 4, 5\}$ using addition and multiplication for all results up to 32.

1	1	12	$(2+1)\cdot 4$	23	$5\cdot 4+2+1$
2	2	13	$2\cdot 4+5$	24	$(5+1)\cdot 4$
3	$2+1$	14	$2\cdot 5+4$	25	$(4+1)\cdot 5$
4	4	15	$(1+2)\cdot 5$	26	$(5+1)\cdot 4+2$
5	5	16	$(5+1)\cdot 2+4$	27	$(4+1)\cdot 5+2$
6	$4+2$	17	$(2+1)\cdot 4+5$	28	$(2+5)\cdot 4$
7	$2+5$	18	$(4+5)\cdot 2$	29	$(5+2)\cdot 4+1$
8	$4\cdot 2$	19	$(5+4)\cdot 2+1$	30	$(2+4)\cdot 5$
9	$5+4$	20	$5\cdot 4$	31	$(4+2)\cdot 5+1$
10	$2\cdot 5$	21	$5\cdot 4+1$	32	$(5+2+1)\cdot 4$
11	$2+4+5$	22	$4\cdot 5+2$		

Here are solutions using numbers in the set $\{1,2,3,8,26\}$ using addition and multiplication for all results up to 98.

1	1	34	$26+8$	67	$(26+3)\cdot 2+8+1$
2	2	35	$26+8+1$	68	$(26+8)\cdot 2$
3	3	36	$26+8+2$	69	$(26+8)\cdot 2+1$
4	$3+1$	37	$26+8+3$	70	$(26+8+1)\cdot 2$
5	$3+2$	38	$26+8+3+1$	71	$(26+8)\cdot 2+3$
6	$3\cdot 2$	39	$26+8+3+2$	72	$(2+1)\cdot 8\cdot 3$
7	$3\cdot 2+1$	40	$(3+2)\cdot 8$	73	$(26+8+1)\cdot 2+3$
8	8	41	$(3+2)\cdot 8+1$	74	$8\cdot 3\cdot 2+26$
9	$8+1$	42	$8\cdot 2+26$	75	$8\cdot 3\cdot 2+26+1$
10	$8+2$	43	$8\cdot 2+26+1$	76	$8\cdot 3+26\cdot 2$
11	$8+3$	44	$(8+1)\cdot 2+26$	77	$8\cdot 3+26\cdot 2+1$
12	$8+3+1$	45	$8\cdot 2+26+3$	78	$26\cdot 3$
13	$8+3+2$	46	$8\cdot 2+26+3+1$	79	$26\cdot 3+1$
14	$3\cdot 2+8$	47	$(8+1)\cdot 2+26+3$	80	$26\cdot 3+2$
15	$3\cdot 2+8+1$	48	$8\cdot 3\cdot 2$	81	$(26+1)\cdot 3$
16	$8\cdot 2$	49	$8\cdot 3\cdot 2+1$	82	$(3\cdot 2+1)\cdot 8+26$
17	$8\cdot 2+1$	50	$8\cdot 3+26$	83	$(26+1)\cdot 3+2$
18	$(8+1)\cdot 2$	51	$8\cdot 3+26+1$	84	$(26+2)\cdot 3$
19	$8\cdot 2+3$	52	$26\cdot 2$	85	$(26+2)\cdot 3+1$
20	$8\cdot 2+3+1$	53	$26\cdot 2+1$	86	$26\cdot 3+8$
21	$(8+1)\cdot 2+3$	54	$(26+1)\cdot 2$	87	$26\cdot 3+8+1$
22	$(8+3)\cdot 2$	55	$26\cdot 2+3$	88	$26\cdot 3+8+2$
23	$(8+3)\cdot 2+1$	56	$26\cdot 2+3+1$	89	$(26+1)\cdot 3+8$
24	$8\cdot 3$	57	$(26+1)\cdot 2+3$	90	$(3+1)\cdot 8\cdot 2+26$
25	$8\cdot 3+1$	58	$(26+3)\cdot 2$	91	$(26+1)\cdot 3+8+2$
26	26	59	$(26+3)\cdot 2+1$	92	$(26+2)\cdot 3+8$
27	$26+1$	60	$26\cdot 2+8$	93	$(26+2)\cdot 3+8+1$
28	$26+2$	61	$26\cdot 2+8+1$	94	$26\cdot 3+8\cdot 2$
29	$26+3$	62	$(26+1)\cdot 2+8$	95	$26\cdot 3+8\cdot 2+1$
30	$26+3+1$	63	$26\cdot 2+8+3$	96	$(8+1)\cdot 2+26\cdot 3$
31	$26+3+2$	64	$(3+1)\cdot 8\cdot 2$	97	$(26+1)\cdot 3+8\cdot 2$
32	$3\cdot 2+26$	65	$(26+1)\cdot 2+8+3$	98	$(2+1)\cdot 8\cdot 3+26$
33	$3\cdot 2+26+1$	66	$(26+3)\cdot 2+8$		