Pick's Theorem

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Part I
Examples

Pick's Theorem provides a method to calculate the area of simple polygons whose vertices lie on lattice points—points with integer coordinates in the x-y plane. The word "simple" in "simple polygon" only means that the polygon has no holes, and that its edges do not intersect. The polygons in Figure 1 are all simple, but keep in mind that the word "simple" may apply only in a technical sense—a simple polygon could have a million edges!

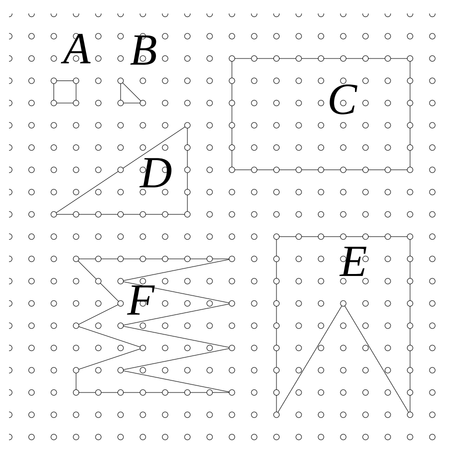

Figure 1: Pick's Theorem Examples

Obviously for polygons with a large interior, the area is going to be roughly approximated by the number of lattice points in the interior. You might guess that a slightly better approximation can be gotten by adding about half the lattice points on the boundary since they are sort of half inside and half outside the polygon. But let's look at a few examples in Figure 1.

For all the examples below, we'll let I be the number of interior vertices, and B be the number of boundary vertices. We will use the notation $\mathcal{A}(P)$ to indicate the area of polygon P.

A: $I = 0$, $B = 4$, $\mathcal{A}(A) = 1$, $I + B/2 = 2$.

B: $I = 0$, $B = 3$, $\mathcal{A}(B) = 1/2$, $I + B/2 = 3/2$.

C: $I = 28$, $B = 26$, $\mathcal{A}(C) = 40$, $I + B/2 = 41$.

D: $I = 7$, $B = 12$. $\mathcal{A}(D) = 12$, $I + B/2 = 13$.

E: It is a bit trickier to calculate the areas of polygons E and F. E can be broken into a 6×3 rectangle and two identical triangles with base 3 and height 5, so we get $I = 22$, $B = 24$ $\mathcal{A}(E) = 33$, $I + B/2 = 34$.

F: It is even uglier to calculate the area for this one, but after some addition and subtraction of areas, we find that: $I = 9$, $B = 26$, $\mathcal{A}(F) = 21$, $I + B/2 = 22$.

What is amazing is that if you look at all six examples above, the estimate $I + B/2$ is *always* off by exactly one. It appears that for any lattice polygon P, the following formula holds exactly:

$$\mathcal{A}(P) = I_p + B_p/2 - 1,$$

where I_p is the number of lattice points completely interior to P and B_p is the number of lattice points on the boundary of P.

This is called Pick's Theorem.

Try a few more examples before continuing.

Part II

Pick's Theorem for Rectangles

Rather than try to do a general proof at the beginning, let's see if we can show that Pick's Theorem is true for some simpler cases. The easiest one to look at is lattice-aligned rectangles.

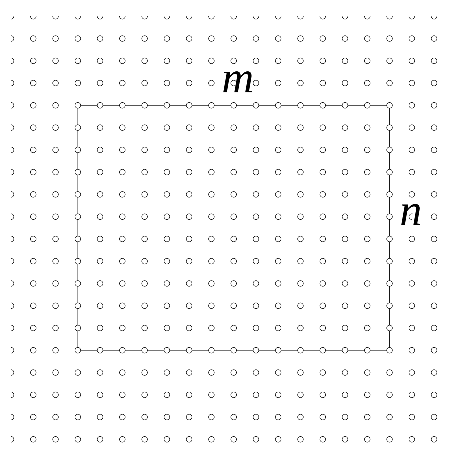

Figure 2: Pick's Theorem for Rectangles

The particular rectangle in Figure 2 is 14×11 ($m = 14$ and $n = 11$), so it has area $14 \cdot 11 = 154$. And it's easy to count the interior and boundary points—there are

$13 \times 10 = 130$ interior points, and there are 50 boundary points. $I + B/2 - 1 = 130 + 50/2 - 1 = 154$, so for this particular rectangle, Pick's Theorem holds.

But what about an arbitrary $m \times n$ rectangle? The area is clearly mn, and it's easy to convince yourself (by drawing a few examples, if necessary), that the number of interior points is given by $I = (m-1)(n-1)$. You can also see that $B = 2m + 2n$ (why?), so for an $m \times n$ rectangle:

$$\begin{aligned} I + B/2 - 1 &= (m-1)(n-1) + 2(m+n)/2 - 1 \\ &= (mn - m - n + 1) + (m+n) - 1 \\ &= mn, \end{aligned}$$

which is exactly what we wanted to show.

Part III

Lattice-Aligned Right Triangles

Just slightly harder is to show that the formula holds for right triangles where the legs of the triangle lie along lattice lines. The easiest way to show this is to think of such a triangle as half of one of the rectangles in the previous part where a diagonal is added. Some examples appear in Figure 3.

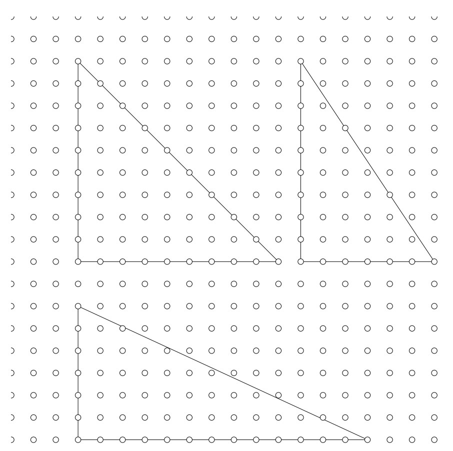

Figure 3: Pick's Theorem for Right Triangles

We will look at such a triangle T with legs of length m and n. The area is clearly $mn/2$, but how many interior and boundary points are there? As you can see from Figure 3, it is easy to count the boundary vertices along the legs, but sometimes the diagonal hits lots of lattice points, sometimes none, and sometimes it hits just a few of them.

But it turns out that it doesn't matter. For an arbitrary right triangle with legs of lengths m and n and area $mn/2$, suppose there are k points on the diagonal, not counting those on the ends (the triangle vertices). The number of boundary points is $m + n + 1 + k$ (why?).

The number of interior points is also easy to calculate. Before you added the diagonal to the rectangle, there were $(m-1)(n-1)$ interior points. If you subtract from this the k points on the boundary, the remainder are split into two halves by the diagonal, so in total the triangle has $((m-1)(n-1)-k)/2$ interior points.

Checking Pick's Theorem for a right triangle with lattice-aligned legs, we get:

$$\begin{aligned} I + B/2 - 1 &= \frac{(m-1)(n-1)-k}{2} + \frac{m+n+1+k}{2} - 1 \\ &= \frac{mn}{2} - \frac{m}{2} - \frac{n}{2} + \frac{1}{2} - \frac{k}{2} + \frac{m}{2} + \frac{n}{2} + \frac{1}{2} + \frac{k}{2} - 1 \\ &= \frac{mn}{2} = \mathcal{A}(T). \end{aligned}$$

Part IV

Pick's Theorem for General Triangles

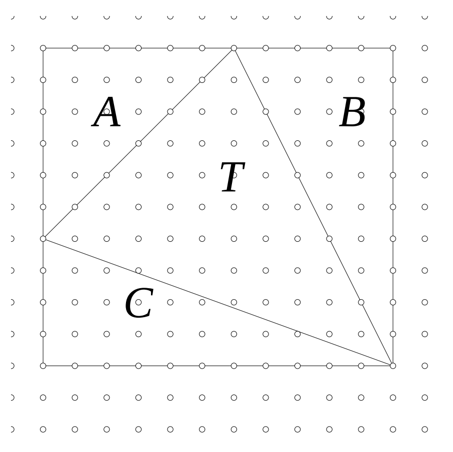

Figure 4: Pick's Theorem for Triangles

Assuming that we know that Pick's Theorem works for right triangles and for rectangles, we can show that it works for arbitrary triangles. In reality there are a bunch of cases to consider, but they all look more or less like variations of Figure 4, where there is an arbitrary triangle T that can be extended to a rectangle with the addition of a few right triangles. In the case of this figure, three additional right triangles are required: A, B, and C.

Suppose that triangle A has I_a interior points and B_a boundary points, that triangle B has I_b interior points and B_b boundary points, et cetera. Call the rectangle R, and let R have I_r interior and B_r boundary points. Since we know that Pick's Theorem works for right triangles and rectangles we have:

$$\begin{aligned}
\mathcal{A}(A) &= I_a + B_a/2 - 1 \\
\mathcal{A}(B) &= I_b + B_b/2 - 1 \\
\mathcal{A}(C) &= I_c + B_c/2 - 1 \\
\mathcal{A}(R) &= I_r + B_r/2 - 1.
\end{aligned}$$

We want to show that

$$\mathcal{A}(T) = I_t + B_t/2 - 1.$$

We know that

$$\begin{aligned}
\mathcal{A}(T) &= \mathcal{A}(R) - \mathcal{A}(A) - \mathcal{A}(B) - \mathcal{A}(C) && (1) \\
&= I_r - I_a - I_b - I_c + (B_r - B_a - B_b - B_c)/2 + 2. && (2)
\end{aligned}$$

Suppose that the rectangle R is $m \times n$ (so it has area $\mathcal{A}(R) = mn$, has $B_r = 2m + 2n$, and has $I_r = (m-1)(n-1)$). If we count the boundary points carefully, we have:

$$B_a + B_b + B_c = B_r + B_t,$$

or

$$B_r = B_a + B_b + B_c - B_t, \quad (3)$$

since the acute-angled vertices of the surrounding triangles are double-counted on both sides of the equation.

Counting (again carefully) the interior points of the rectangle, we get:

$$I_r = I_a + I_b + I_c + I_t + (B_a + B_b + B_c - B_r) - 3. \quad (4)$$

We need the final -3 in the equation above because now the corners of the triangles really are double-counted.

Substitute the value of B_r in Equation 3 into Equation 4:

$$I_r = I_a + I_b + I_c + I_t + B_t - 3. \quad (5)$$

Now we just substitute the values for B_r and I_r from Equations 3 and 5 into Equation 2 to obtain (after a bit of algebra):

$$\begin{aligned}
\mathcal{A}(T) &= I_r - I_a - I_b - I_c + (B_r - B_a - B_b - B_c)/2 + 2 \\
&= (I_a + I_b + I_c + I_t + B_t - 3) - I_a - I_b - I_c \\
&\quad + ((B_a + B_b + B_c - B_t) - B_a - B_b - B_c)/2 + 2 \\
&= I_t + B_t - 3 - B_t/2 + 2 \\
&= I_t + B_t/2 - 1,
\end{aligned}$$

which is exactly what we wanted to show.

If you would like to check the equations above with the example in Figure 4, here are the values for all four triangles and the rectangle:

	I	B	$\mathcal{A} = I + B/2 - 1$
A	10	18	$18 = 10 + 18/2 - 1$
B	16	20	$25 = 16 + 20/2 - 1$
C	15	16	$22 = 15 + 16/2 - 1$
T	40	12	$45 = 40 + 12/2 - 1$
R	90	42	$110 = 90 + 42/2 - 1$

Part V

Pick's Theorem: General Case

We now know that Pick's Theorem is true for arbitrary triangles with their vertices on lattice points. (Well, we do if we have checked a few additional cases that are similar to the one that appears in the previous section.) How do we show that it is true for an arbitrary simple polygon P with vertices that are lattice points?

1 Overview of the Proof

Intuitively, what we will do is show that any such polygon can be constructed by putting together smaller polygons where we know that Pick's theorem is true. Roughly, we will go about it as follows. We have already shown that every 3-sided lattice polygon satisfies Pick's Theorem. Next, we show that if it's true for all 3-sided polygons, it is also true for all 4-sided polygons. Then we show that if it's true for all 3- and 4-sided polygons, it is true for all 5-sided polygons. Then we show that if it's true for all 3-, 4-, and 5-sided polygons, it is true for all 6-sided polygons, et cetera.

This technique is officially known as generalized mathematical induction, and we will not, in practice, do all of the infinite number of steps that we began to describe in the previous paragraph. We will do the proof in two steps, the first of which has already been completed:

1. Show that the theorem is true for every lattice polygon having 3 sides.

2. Show that if the theorem is true for every lattice polygon having 3 or 4 or 5 or $\ldots$ or $k-1$ sides, then it is true for every lattice polygon having k sides.

Since the second part of the proof works for *any* k, it effectively amounts to doing all of the infinite number of steps listed a couple of paragraphs previously.

A particular example of the general idea is illustrated in Figure 5. We have a 23-sided polygon: $ABC \cdots W$. We will show that every such polygon with more than three sides has an interior diagonal (there are lots of examples in this figure, but we have chosen diagonal OW as an example), and such a diagonal will split the polygon into a pair of smaller polygons. In this case, into the 16-sided polygon $ABC \cdots MNOW$ and the 9-sided polygon $OPQ \cdots W$. Since we are this stage in the proof, we know Pick's Theorem is true for all polygons having between 3 and 22 edges, then in particular it will be true for the 16- and 9-sided polygons above. We then show that if two

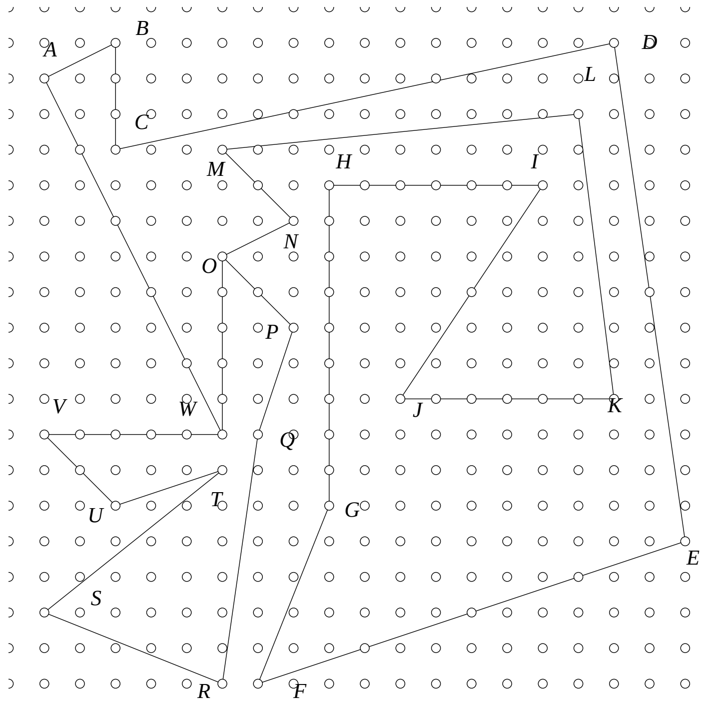

Figure 5: Pick's Theorem: General Case

such polygons that satisfy Pick's Theorem are attached together, the resulting polygon will also satisfy Pick's Theorem.

We will show the second part first—that if two polygons satisfy the theorem, then the joined version will also satisfy the theorem.

2 Joining Two Lattice Polygons

Suppose the two sub-polygons of the original polygon P are P_1 and P_2, where P_1 has I_1 interior points and B_1 boundary points. P_2 has I_2 interior and B_2 boundary points. Let's also assume that the common diagonal of the original polygon between P_1 and P_2 contains m points. Let P have I interior and B boundary points.

$$\mathcal{A}(P) = \mathcal{A}(P_1) + \mathcal{A}(P_2) = (I_1 + B_1/2 - 1) + (I_2 + B_2/2 - 1).$$

Since any point interior to P_1 or P_2 is interior to P, and since $m - 2$ of the common boundary points of P_1 and P_2 are also interior to P, $I = I_1 + I_2 + m - 2$. Similar reasoning gives $B = B_1 + B_2 - 2(m - 2) - 2$.

Therefore:

$$I + B/2 - 1 \quad = \quad (I_1 + I_2 + m - 2) + (B_1 + B_2 - 2(m - 2) - 2)/2 - 1$$

$$\begin{aligned} &= (I_1 + B_1/2 - 1) + (I_2 + B_2/2 - 1) \\ &= \mathcal{A}(P). \end{aligned}$$

3 The Interior Diagonal

To complete the proof of Pick's Theorem, we must show that any simple polygon has an interior diagonal—a diagonal completely within the polygon that connects two of its vertices.

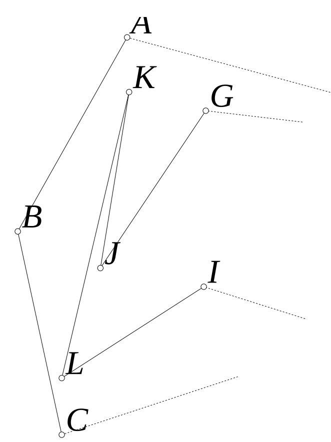

Figure 6: Existence of an Interior Diagonal

The proof goes as follows. Find an angle $\angle ABC$ such that the interior of the polygon is on the side of the angle less than $180°$. Then there are two cases. Either the line segment AC lies completely within the polyon in which case we are done, and AC is the required diagonal, or some part of the polygon (shown as $GJKL$ in Figure 6) goes inside $\triangle ABC$.

There are only a finite number of vertices of the polygon interior to $\triangle ABC$; for each of those, construct a line perpendicular to the angle bisector of $\angle ABC$. Clearly, the line connecting B to the vertex with perpendicular closest to point B will lie completely within the polygon. If not, it had to cross another edge of the polygon, and one end of that edge would have a perpendicular to the angle bisector closer to B.

Note that we *cannot* use the vertex closest to B. In Figure 6, J is the point nearest B, but clearly segment JB crosses segment KL.

Part VI

Polygons with Holes

So far, all of the polygons we have considered are simple—they have no holes. In figure 7 are five examples of polygons with holes. Polygons A, B and C have a single

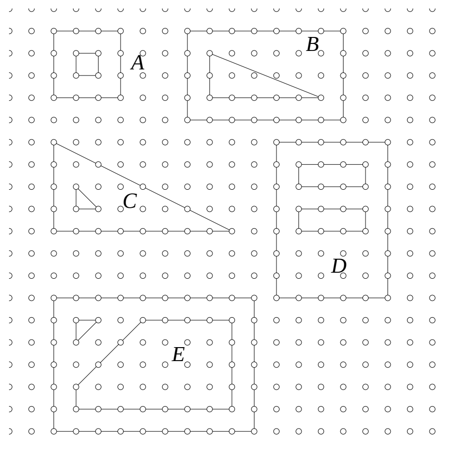

Figure 7: Polygons with holes

hole, while polygons D and E each have two holes. These examples are simple enough that it is not difficult to calculate the areas enclosed inside the outer polygon and outside the hole or holes.

	I	B	$\mathcal{A}$	$I + B/2 - 1$
A	0	16	8	7
B	8	30	23	22
C	6	19	15.5	14.5
D	8	40	29	27
E	3	52	30	28

The table above shows the counts, including the actual area and the area predicted by the formula that works for polygons without holes. In the cases where there is one hole, there is an error of 1; in the cases where there are two holes, the error is 2. In fact, if you try a few more examples with one, two or more holes and add additional entries to the table above, you will find that the area seems to be given by the following formula, where n is the number of holes:

$$\mathcal{A} = I + B/2 - 1 + n.$$

Since we already know the formula for the areas of polygons without holes, we can use that information to work out the area of a polygon with holes. We will first work out the formula for the area of a polygon with a single hole and then we will extend that calculation to the general case of n holes.

For the single hole case, we need to show that the area of the polygon is given by the formula $I + B/2 - 1 + 1 = I + B/2$.

Suppose the outer polygon has area $\mathcal{A}_o$, has I_o interior points and B_o boundary points. The polygon that makes up the hole has area $\mathcal{A}_h$ and has I_h interior points and B_h boundary points.

From our previous work, we know that $\mathcal{A}_o = I_o + B_o/2 - 1$ and $\mathcal{A}_h = I_h + B_h/2 - 1$. Since $\mathcal{A} = \mathcal{A}_o - \mathcal{A}_h$, we have

$$\mathcal{A} = I_o - I_h + (B_o - B_h)/2.$$

If I and B are the number of interior and boundary points of the entire polygon that includes the hole, we have $I = I_o - I_h - B_h$ and $B = B_o + B_h$. Using these formulas and a little algebra, we obtain:

$$I + B/2 = I - I_h + (B_o - B_h)/2 = \mathcal{A},$$

which is exactly what we were trying to prove.

4 The General Case with n Holes

A similar calculation can be made for polygons with any number of holes. As before, let $\mathcal{A}_o$ be the area of the outer polygon (with I_o and B_o interior and boundary points) have n holes with areas $\mathcal{A}_\rangle$, $1 \leq i \leq n$ and having I_i and B_i interior and boundary points.

The area $\mathcal{A}$ of the "holey" polygon (having I and B interior and boundary points) is given by:

$$\mathcal{A} = \mathcal{A}_o - \sum_{i=1}^{n} \mathcal{A}_i.$$

Since $\mathcal{A}_o = I_o + B_o/2 - 1$ and $\mathcal{A}_i = I_i + B_i/2 - 1$ we have:

$$\begin{aligned}
\mathcal{A} &= I_o + B_o/2 - 1 - \sum_{i=1}^{n}(I_i + B_i/2 - 1) \\
&= I_o + B_o/2 - 1 + n - \sum_{i=1}^{n}(I_i + B_i/2).
\end{aligned}$$

It is easy to see that $I = I_o - \sum_{i=i}^{n}(I_i + B_i)$ and that $B = B_o + \sum_{i=i}^{n} B_i$. Thus:

$$I + B/2 - 1 + n = I_o + B_o/2 - \sum_{i=1}^{n}(I_i + B_i/2) - 1 + n = \mathcal{A},$$

which is exactly what we were trying to prove.