Exploring Pascal's Triangle

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Abstract

This article provides material to help a teacher lead a class in an adventure of mathematical discovery using Pascal's triangle and various related ideas as the topic. There is plenty of mathematical content here, so it can certainly be used by anyone who wants to explore the subject, but pedagogical advice is mixed in with the mathematics.

1 General Hints for Leading the Discussion

The material here should not be presented as a lecture. Begin with a simple definition of the triangle and have the students look for patterns. When they notice patterns, get them to find proofs, when possible. By "proof" we do not necessarily mean a rigorous mathematical proof, but at least enough of an argument that it is convincing and that could, in principle, be extended to a rigorous proof. Some sample arguments/proofs are presented below, but they represent only one approach; try to help the students find their own way, if possible.

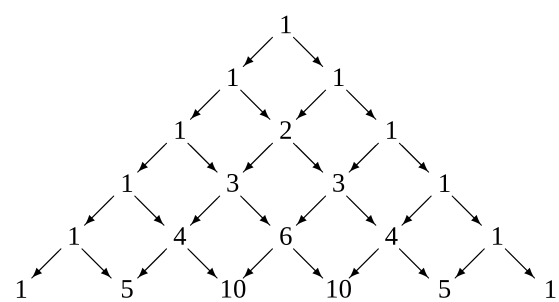

Figure 1: Pascal's Triangle

It is *not* critical to cover all the topics here, or to cover them in any particular order, although the order below is reasonable. It is important to let the investigation continue in its own direction, with perhaps a little steering if the class is near something very interesting, but not quite there.

The numbers in Pascal's triangle provide a wonderful example of how many areas of mathematics are intertwined, and how an understanding of one area can shed light on other areas. The proposed order of presentation below shows how real mathematics research is done: it is not a straight line; one bounces back and forth among ideas, applying new ideas back to areas that were already covered, shedding new light on them, and possibly allowing new discoveries to be made in those "old" areas.

Finally, the material here does not have to be presented in a single session, and in fact, multiple sessions might be the most effective presentation technique. That way there's some review, and the amount of new material in each session will not be overwhelming.

2 Basic Definition of Pascal's Triangle

Most people are introduced to Pascal's triangle by means of an arbitrary-seeming set of rules. Begin with a 1 on the top and with 1's running down the two sides of a triangle as in figure 1. Each additional number lies between two numbers and below them, and its value is the sum of the two numbers above it. The theoretical triangle is infinite and continues downward forever, but only the first 6 lines appear in figure 1. In the figure, each number has arrows pointing to it from the numbers whose sum it is. More rows of Pascal's triangle are listed in Appendix B.

A different way to describe the triangle is to view the first line is an infinite sequence of zeros except for a single 1. To obtain successive lines, add every adjacent pair of numbers and write the sum between and below them. The non-zero part is Pascal's triangle.

3 Some Simple Observations

Now look for patterns in the triangle. We're interested in everything, even the most obvious facts. When it's easy to do, try to find a "proof" (or at least a convincing argument) that the fact is true. There are probably an infinite number of possible results here, but let's just look at a few, including some that seem completely trivial. In the examples below, some typical observations are in bold-face type, and an indication of a proof, possibly together with additional comments, appears afterwards in the standard font.

All the numbers are positive. We begin with only a positive 1, and we can only generate numbers by including additional 1's, or by adding existing positive numbers. (Note that this is really an inductive proof, if written out formally.)

The numbers are symmetric about a vertical line through the apex of the triangle. The initial row with a single 1 on it is symmetric, and we do the same things on both sides, so however a number was generated on the left, the same thing was done to obtain the corresponding number on the right. This is a fundamental idea in mathematics: if you do the same thing to the same objects, you get the same result.

Look at the patterns in lines parallel to the edges of the triangle. There are nice patterns. The one that is perhaps the nicest example is the one that goes:

$$1, 3, 6, 10, 15, 21, \ldots$$

These are just the sums: (1), $(1 + 2)$, $(1 + 2 + 3)$, $(1 + 2 + 3 + 4)$, et cetera. A quick examination shows why the triangle generates these numbers. Note that they are sometimes called "triangular numbers" since if you make an equilateral triangle of coins, for example, these numbers count the total number of coins in the triangle. In fact, the next row:

$$1, 4, 10, 20, 35, \ldots$$

are called the "pyramidal numbers". They would count the number of, say, cannonballs that are stacked in triangular pyramids of various sizes. Is it clear why adding triangular numbers together give the pyramidal numbers? Is it clear how Pascal's triangle succeeds in adding the triangular numbers in this way? In the same vein, if those rows represent similar counts in 2 and 3 dimensions, shouldn't the first two rows somehow represent counts of something in 0 and 1 dimensions? They do – and this is could be a nice segue into the behavior of patterns in 4 and higher dimensions.

There is another "application" of this fourth diagonal. They are the sums of the triangular numbers, and if you think about the song, "The Twelve Days of Christmas", the triangular numbers are the number of gifts given by our true love on each day, so the sum of the triangular numbers is the total numbers of items given up to that day. Later in this document we shall derive formulas for the elements in the triangle, and a trivial calculation would tell us that after the twelfth day of Christmas, we would have received from our true love a total of 364 items.

If you add the numbers in a row, they add to powers of 2. If we think about the rows as being generated from an initial row that contains a single 1 and an infinite number of zeroes on each side, then each number in a given row adds its value down both to the right and to the left, so effectively two copies of it appear. This means that whatever sum you have in a row, the next row will have a sum that is double the previous. It's also good to note that if we number the rows beginning with row 0 instead of row 1, then row n sums to 2^n. This serves as a nice reminder that $x^0 = 1$, for positive numbers x.

If you alternate the signs of the numbers in any row and then add them together, the sum is 0. This is easy to see for the rows with an even number of terms, since some quick experiments will show that if a number on the left is positive, then the symmetric number on the right will be negative, as in: $1 - 5 + 10 - 10 + 5 - 1$. One way to see this is that the two equal numbers in the middle will have opposite signs, and then it's easy to trace forward and back and conclude that every symmetric pair will have opposite signs.

It's worth messing around a bit to try to see why this might work for rows with an odd number. There are probably lots of ways to do it, but here's a suggestion. Look at a typical row, like the fifth:

$$+1\ -5\ +10\ -10\ +5\ -1.$$

We'd like the next row (the sixth, in this case) to look like this:

$$+1\ -6\ +15\ -20\ +15\ -6\ +1.$$

If we give letter names to the numbers in the row above it:

$$a = +1; b = -5; c = +10; d = -10; e = +5; f = -1,$$

then how can we write the elements in row 6 in terms of those in row 5? Here's one nice way to do it:

$$+1 = a - 0; -6 = b - a; +15 = c - b; -20 = d - c; +15 = e - d; -6 = f - e; +1 = 0 - f.$$

Now just add the terms:

$$a - 0 + b - a + c - b + d - c + e - d + f - e + 0 - f,$$

and the sum is obviously zero since each term appears twice, but with opposite signs.

The "hockey-stick rule": Begin from any 1 on the right edge of the triangle and follow the numbers left and down for any number of steps. As you go, add the numbers you encounter. When you stop, you can find the sum by taking a 90-degree turn on your path to the right and stepping down one. It is called the hockey-stick rule since the numbers involved form a long straight line like the handle of a hockey-stick, and the quick turn at the end where the sum appears is like the part that contacts the puck. Figure 2 illustrates two of them. The upper one adds $1 + 1 + 1 + 1 + 1$ to obtain 5, and the other adds $1 + 4 + 10 + 20$ to obtain 35. (Because of the symmetry of Pascal's triangle, the hockey sticks could start from the left edge as well.)

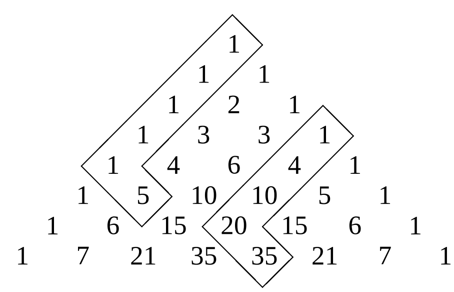

Figure 2: The Hockey Stick

To see why this always works, note that whichever 1 you start with and begin to head into the triangle, there is a 1 in the other direction, so the sum starts out correctly. Then note that the number that sits in the position of the sum of the line is always created from the previous sum plus the new number.

Note how this relates to the triangular and pyramidal numbers. If we think of pyramids as "three-dimensional triangles" and of lines with $1, 2, 3, 4, \ldots$ items in them as "one dimensional triangles", and single items as a "zero-dimensional triangle", then the sum of zero-dimensional triangles make the one dimensional triangles, the sum of the one-dimensional trinagles make the two-dimensional triangles and so on. With this interpretation, look at the diagonals of Pascal's triangle as zero, one, two, three, ... dimensional triangles, and see how the hockey-stick rule adds the items in each diagonal to form the next diagonal in exactly the manner described above.

There are interesting patterns if we simply consider whether the terms are odd or even. See figure 3. In the figure, in place of the usual numbers in Pascal's triangle we have circles that are either black or white, depending upon whether the number in that position is odd or even, respectively.

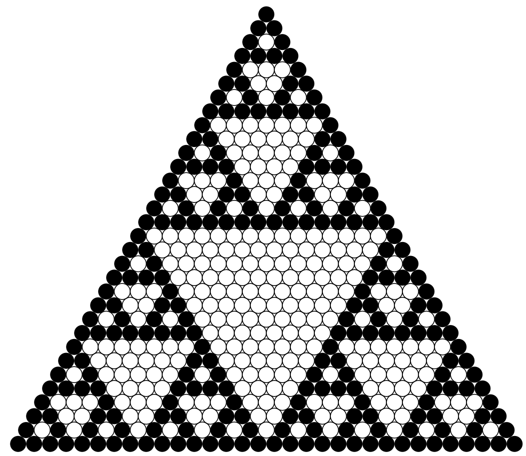

Figure 3: Odd-Even Pascal's Triangle

Look at the general pattern, but it is also interesting to note that certain rows are completely black. What are those row numbers? They are rows $0, 1, 3, 7, 15, 31$, and each of those numbers is one less than a perfect power of 2.

How could you possibly prove this? Well, one approach is basically recursive: Notice the triangles of even numbers with their tips down. Clearly, since adding evens yields an even, the interiors will remain even, but at the edges where they're up against an odd number, the width will gradually decrease to a point. Now look at the little triangle made from the four rows 0 through 3. At the bottom, you've got all odd numbers, so the next line will be all even, except for the other edges. The outer edges must look like two copies of the initial triangle until they meet. Once you've got all odd, we now have the shape of the triangle made of the first 8 rows, and the next step is two odds at the end, with evens solidly between them. The argument repeats, but with triangles of twice the size, et cetera.

There's nothing special about odd-even; the same sorts of investigations can be made looking for multiples of other numbers.

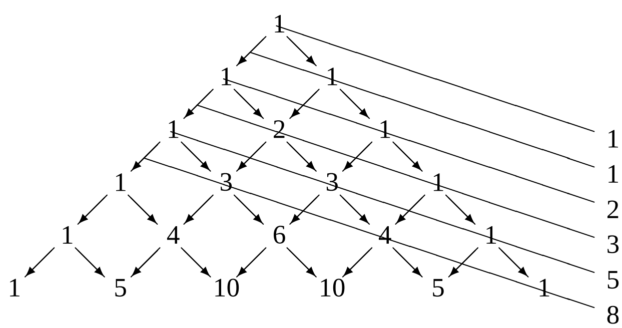

Figure 4: Fibonacci Series

The Fibonacci sequence is hidden in Pascal's triangle.

See figure 4. If we take Pascal's triangle and draw the slanting lines as shown, and add the numbers that intersect each line, the sums turn out to be the values in the Fibonacci series:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots.$$

The first two numbers are 1 and every number after that is simply the sum of the two previous numbers.

One argument to convince yourself that this is true is to note that the first two lines are OK, and then to note that each successive line is made by combining exactly once, each of the numbers on the previous two lines. In other words, note that the sums satisfy exactly the same rules that the Fibonacci sequence does: the first two sums are one, and after that, each sum can be interpreted as the sum of the two previous sums.

4 Pascal's Triangle and the Binomial Theorem

Most people know what happens when you raise a binomial to integer powers. The table below is slightly unusual in that coefficients of 1 are included since it will be the coefficients that are of primary importance in what follows:

$$\begin{aligned}
(L+R)^0 &= 1 \\
(L+R)^1 &= 1L + 1R \\
(L+R)^2 &= 1L^2 + 2LR + 1R^2 \\
(L+R)^3 &= 1L^3 + 3L^2R + 3LR^2 + 1R^3 \\
(L+R)^4 &= 1L^4 + 4L^3R + 6L^2R^2 + 4LR^3 + 1R^4 \\
(L+R)^5 &= 1L^5 + 5L^4R + 10L^3R^2 + 10L^2R^3 + 5LR^4 + 1R^5
\end{aligned}$$

A quick glance shows that the coefficients above are exactly the same as the numbers in Pascal's triangle. If this is generally true, it is easy to expand a binomial raised to an arbitrary power. If we want to deal with $(L+R)^n$, we use as coefficients the numbers in row n of Pascal's triangle. (Note again why it is convenient to assign the first row the number zero.) To the first coefficient, we assign L^n, and for each successive coefficient, we lower the exponent on L and raise the exponent on R. (Note that we could have said, "assign L^nR^0 to the first coefficient.) The exponent on L will reach 0 and the exponent on R will reach n just as we arrive at the last coefficient in row n of Pascal's triangle.

OK, but why does it work? The easiest way to see your way through to a proof is to look at a couple of cases that are not too complex, but have enough terms that it's easy to see patterns. For the example here, we'll assume that we've successfully arrived at the expansion of $(L+R)^4$ and we want to use that to compute the expansion of $(L+R)^5$.

The brute-force method of multiplication from the algebra 1 class is probably the easiest way to see what's going on. To obtain $(L+R)^5$ from $(L+R)^4$, we simply need to multiply the latter by $(L+R)$:

$$\begin{array}{ccccccccccc}
 & & L^4 & + & 4L^3R & + & 6L^2R^2 & + & 4LR^3 & + & R^4 \\
 & & & & & & & & L & + & R \\
\hline
 & & L^4R & + & 4L^3R^2 & + & 6L^2R^3 & + & 4LR^4 & + & R^5 \\
L^5 & + & 4L^4R & + & 6L^3R^2 & + & 4L^2R^3 & + & LR^4 & & \\
\hline
L^5 & + & 5L^4R & + & 10L^3R^2 & + & 10L^2R^3 & + & 5LR^4 & + & R^5
\end{array} \qquad (1)$$

In the multiplication illustrated in equation (1) we see that the expansion for $(L+R)^4$ is multiplied first by R, then by L, and then those two results are added together. Multiplication by R simply increases the exponent on R by one in each term and similarly for multiplication by L. In other words, before the expressions are added, they have the same coefficients; the only thing that has changed are the values of the exponents.

But notice that the two multiplications effectively shift the rows by one unit relative to each other, so when we combine the multiplications of the expansion of $(L+R)^4$ by L and R, we wind up adding adjacent coefficients. It's not too hard to see that this is exactly the same method we used to generate Pascal's triangle.

But once we're convinced that the binomial theorem works, we can use it to re-prove some of the things we noticed in section 3. For example, to show that the numbers in row n of Pascal's triangle add to 2^n, just consider the binomial theorem expansion of $(1+1)^n$. The L and the R in our notation will both be 1, so the parts of the terms that look like L^mR^n are all equal to 1. Thus $(1+1)^n = 2^n$ is the sum of the numbers in row n of Pascal's triangle. Similarly, to show that with alternating signs the sum is zero, look at the expansion of $(1-1)^n = 0^n$.

5 An Application to Arithmetic

A possible introduction to the previous section might be to have the class look at powers of 11:

$$\begin{aligned}
11^0 &= 1\\
11^1 &= 11\\
11^2 &= 121\\
11^3 &= 1331\\
11^4 &= 14641\\
11^5 &= 161051\\
11^6 &= 1771561
\end{aligned}$$

It's interesting that up to the fourth power, the digits in the answer are just the entries in the rows of Pascal's triangle. What is going on, of course, is that $11 = 10 + 1$, and the answers are just $(10 + 1)^n$, for various n. Everything works great until the fifth row, where the entries in Pascal's triangle get to be 10 or larger, and there is a carry into the next row. Although Pascal's triangle is hidden, it does appear in the following sense. Consider the final number, 11^6:

$$\begin{array}{rcrcr}
(10+1)^6 & = & 10^6 & = & 1000000\\
 & + & 6\cdot 10^5 & = & 600000\\
 & + & 15\cdot 10^4 & = & 150000\\
 & + & 20\cdot 10^3 & = & 20000\\
 & + & 15\cdot 10^2 & = & 1500\\
 & + & 6\cdot 10^1 & = & 60\\
 & + & 1\cdot 10^0 & = & 1\\
\hline
 & = & & & 1771561
\end{array}$$

By shifting the columns appropriately, the numbers in any row of Pascal's triangle can be added to calculate 11^n, by using the numbers in row n.

Could similar ideas be used to calculate 101^n or 1001^n?

6 Combinatorial Aspects of Pascal's Triangle

Before going into the theory, it's a good idea to look at a few concrete examples to see how one could do the counting without any theory, and to notice that the counts we obtain from a certain type of problem (called "combinations") all happen to be numbers that we can find in Pascal's triangle.

Let's start with an easy one: How many ways are there to choose two objects from a set of four? It doesn't take too long to list them for some particular set, say $\{A, B, C, D\}$. After a little searching, it appears that this is a complete list:

$$AB, AC, AD, BC, BD, CD.$$

The first time students try to count them, it's unlikely that they'll come up with them in a logical order as presented above, but they'll search for a while, find six, and after some futile searching, they'll be convinced that they've got all of them. The obvious question is, "How do you *know* you've got them all?"

There are various approaches, but one might be something like, "We'll list them in alphabetical order. First find all that begin with A. Then all that begin with B, and so on."

Try a couple of others; say, 3 objects from a set of 5. The set is $\{A, B, C, D, E\}$ and here are the 10 possible groups of objects (listed in alphabetical order):

$$ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE$$

Note that the strategy still works, but we have to be careful since even while we're working on the part where we find all triples that start with A, we still have to find all the pairs that can follow. Note that this has, in a sense, been solved in the previous example, since if you know you're beginning with A, there are four items left, and the previous exercise showed us that there are six ways to do it.

Now count the number of ways to choose 2 items from a set of five. Use the same set: $\{A, B, C, D, E\}$, and here are the 10 results:

$$AB, AC, AD, AE, BC, BD, BE, CD, CE, DE$$

Is it just luck that there are the same number of ways of getting 3 items from a set of 5 and 2 items from a set of 5? A key insight here is that if I tell you which ones I'm *not* taking, that tells you which ones I *am* taking. Thus for each set of 2 items, I can tell you which 2 they are, or, which 3 they aren't! Thus there must be the same number of ways of choosing 2 from 5 or 3 from 5.

Obviously, the same thing will hold for any similar situation: there are the same number of ways to pick 11 things out of 17 as there are to pick 6 out of 17, and so on. This is the sort of thing a mathematician would call "duality". The general statement is this: There are the same number of ways to choose k things from n as there are to choose $n - k$ things from n, assuming that $k \leq n$.

After you've looked at a few simple situations, it's easy to get a lot of other examples. The easiest is: How many ways are there to pick 1 item from a set of n? The answer is obviously n. And from the previous paragraphs, there are also n ways to choose $n - 1$ items from a set of n.

A slightly more difficult concept is involved in the answer to this question: How many ways are there to choose 0 (zero) items from a set of n? The correct answer is always 1 – there is a single way to do it: just pick nothing. Or another way to look at it is that there's clearly only one way to choose all n items from a set of n: take all of them. But the duality concept that we've just considered would imply that there are the same way to choose n items from n as 0 items from n.

After looking at a few of these, we notice that the counts we obtain are the same as the numbers we find in Pascal's triangle. Not only that, but, at least for the few situations we've looked at, the number of ways to choose k things from a set of n seems to be the number in column k (starting the column count from zero) and in row n (again, starting the row count from zero). The only entry that might seem a little strange is the one for row zero, column zero, but even then, it ought to be 1, since there's really only one way to choose no items from an empty set: just take nothing.

With this encouragement, we can try to see why it might be true that combinations and the numbers in Pascal's triangle are the same.

First, a little notation. In order to avoid saying over and over something like, "the number of ways to choose k objects from a set of n objects", we will simply say "n choose k". There are various ways to write it, but "(n choose k)" works, with the parentheses indicating a grouping. The most common form, of course, is that of the binomial coefficient: $\binom{n}{k}$, which will turn out to be the same thing. So from our previous work, we can say that (5 choose 2) = (5 choose 3) = 10, or, alternatively, $\binom{5}{2} = \binom{5}{3} = 10$.

Here's one way to look at it: We'll examine a special case and see why it works. Then, if we look at the special numbers we've chosen, we'll see that there is nothing special at all about them, and the general case is just a particular example.

Suppose we need to find out how many ways there are to choose 4 things from a set of 7, and let's say that we've already somehow worked out the counts for all similar problems for sets containing 6 or fewer objects. For concreteness, let's say that the set of 7 things is $\{A, B, C, D, E, F, G\}$. If we consider the sets of four items that we can make, we can divide them into two groups. Some of them will contain the member A (call this group 1) and some will not (group 2).

Every one of the sets in group 1 has an A plus three other members. Those additional three members must be chosen from the set $\{B, C, D, E, F, G\}$ which has six elements. There are (6 choose 3) ways to do this, so there are (6 choose 3) elements in group 1. In group 2, the element A does not appear, so the elements of group 2 are all the ways that you can choose 4 items from a set of the remaining 6 objects. Thus there are (6 choose 4) ways to do this. Thus:

$$(7 \text{ choose } 4) = (6 \text{ choose } 3) + (6 \text{ choose } 4)$$

or, using the binomial coefficients:

$$\binom{7}{4} = \binom{6}{3} + \binom{6}{4}.$$

Now there's clearly nothing special about 7 and 4. To work out the value of (n choose k) we pick one particular element and divide the sets into two classes: one of subsets containing that element and the other of subsets that do not. There are ($n - 1$ choose $k - 1$) ways to choose subsets of the first type and ($n - 1$ choose k) ways to choose subsets of the second type. Add them together for the result:

$$(n \text{ choose } k) = (n - 1 \text{ choose } k - 1) + (n - 1 \text{ choose } k)$$

or:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

If we map these back to Pascal's triangle, we can see that they amount *exactly* to our method of generating new lines from previous lines.

7 Back to the Binomial Theorem

Now, let's go back to the binomial theorem and see if we can somehow interpret it as a method for choosing "k items from a set of n".

Multiplication over the real numbers is commutative, in the sense that $LR = RL$ – we can reverse the order of a multiplication and the result is the same. If we were to do a multiplication of a binomial by itself in a strictly formal way, the steps would look like this:

$$\begin{aligned}
(L + R)(L + R) &= L(L + R) + R(L + R) \\
&= LL + LR + RL + RR \\
&= LL + LR + LR + RR \\
&= LL + 2LR + RR.
\end{aligned}$$

The first step uses the distributive law; the next uses the distributive law again, then we use the commutative law of multiplication to change the RL to LR, and finally, we can combine the two copies of LR to obtain the product in the usual form – well, usual except that we've written LL and RR instead of L^2 and R^2 for reasons that will become clear later.

But suppose for a minute that we cannot use the commutative law of multiplication (but that we can rearrange the terms, so that addition *is* commutative). Using the distributive law we can still do all the multiplications needed to generate $(L+R)^n$, but we will wind up with a lot of terms that cannot be combined. In fact, none of them can be combined, and $(L+R)^n$ will contain 2^n terms. We can compute $(L+R)^{n+1}$ by multiplying out the expanded form of $(L+R)^n$ by one additional $(L+R)$. The calculation above shows the result of $(L+R)^2$; we'll use that to generate $(L+R)^3$:

$$\begin{aligned}(L+R)^3 &= (L+R)(L+R)^2 \\ &= (L+R)(LL+LR+RL+RR) \\ &= L(LL+LR+RL+RR)+R(LL+LR+RL+RR) \\ &= LLL+LLR+LRL+LRR+RLL+RLR+RRL+RRR.\end{aligned}$$

Without going through the detailed calculations that we used above, but using the same method, here is what we would obtain for $(L+R)^4$:

$$\begin{aligned}(L+R)^4 = \; & LLLL+LLLR+LLRL+LLRR+LRLL+LRLR+LRRL+LRRR+ \\ & RLLL+RLLR+RLRL+RLRR+RRLL+RRLR+RRRL+RRRR.\end{aligned}$$

Notice that in our expansions in this manner of $(L+R)^2$, $(L+R)^3$ and $(L+R)^4$, the results are simply all possible arrangements of 2, 3 or 4 R's and L's. It's easy to see why. If we multiply out something like:

$$(L+R)(L+R)(L+R)(L+R)$$

we are basically making every possible choice of one of the two in each set of parentheses, and since there are 2 choices per group and 4 groups, there are $2^4 = 16$ possible sets of choices.

Now, when we do have commutativity, we convert terms like $RLRL$ to L^2R^2 throughout, and then combine like terms. Let's do that, but in the opposite order: first, we'll combine the terms we know will result in the same value, as shown below. Groups with the same number of R's and L's are enclosed in parentheses:

$$\begin{aligned}(L+R)^2 &= (LL)+(LR+RL)+(RR) \\ (L+R)^3 &= (LLL)+(LLR+LRL+RLL)+(LRR+RLR+RRL)+(RRR) \\ (L+R)^4 &= (LLLL)+(LLLR+LLRL+LRLL+RLLL)+ \\ & \quad (LLRR+LRLR+LRRL+RLLR+RLRL+RRLL)+ \\ & \quad (LRRR+RLRR+RRLR+RRRL)+(RRRR).\end{aligned}$$

The groups above have sizes: $[1,2,1]$, then $[1,3,3,1]$, then $[1,4,6,4,1]$. These are the numbers in rows 2, 3 and 4 of Pascal's triangle. Stop for a second and look closely at these grouped terms to see if there is some way to interpret them as (n choose k).

Here is one way. Look at the largest group: the six terms with 2 R's and 2 L's in the expansion of $(L+R)^4$:

$$(LLRR+LRLR+LRRL+RLLR+RLRL+RRLL).$$

If we interpret the four letters as indicating positions of four items in a set, then an L means "choose the item" and an R means "do not choose it". Thus $LLRR$ means to take the first two and omit the second two; $RLLR$ means to take the second and third items only, and so on.

Clearly, since all the possibilities appear here, the number of terms (6) is exactly the same as the number of ways that we can choose 2 items from a set of 4. When the commutative law of addition is applied to these terms, since they all have 2 R's and 2 L's, all will become L^2R^2, and since there are 6 of them, the middle term of the expansion of $(L+R)^4$ will be $6L^2R^2$.

Again, there's nothing special about the middle term of the expansion using the fourth power; the same arguments can be used to show that *every* term in *every* binomial expansion can be interpreted in its combinatorial sense.

8 Statistics: The Binomial Distribution

In the previous section (Section 7) we looked at the patterns of L and R that related to raising a binomial to a power: $(L+R)^n$. If we are instead looking at a game that consists of flipping a coin n times, and are interested in the patterns of "heads" and "tails" that could arise, it will turn out that if we just substitute "T" for "L" and "H" for "R" then we will have basically described the situation.

Consider flipping a fair coin (a coin that has equal chances of landing "heads" or "tails" which we will denote from now on as "H" and "T") 3 times. If we indicate the result of such an experiment as a three-letter sequence where the first is the result of the first flip, the second represents the second, and so on, then here are all the possibilities:

$$HHH,\ HHT,\ HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT. \tag{2}$$

Note that this (except for the "$+$" signs) is exactly what we would get if we do the multiplication below without having the commutative law as we did in the expansions of $(L+R)^3$ and $(L+R)^4$ in Section 7:

$$(H+T)^3 = HHH + HHT + HTH + HTT + THH + THT + TTH + TTT.$$

There is no reason to believe, since the coin is fair, that any of the patterns in 2 is any more or less likely than any other, and if the only thing you are interested in is the number of "heads", then you can see that HHH and TTT both occur once, while results with one "head" occurs three times and similarly for results with two "heads". Thus, if you were to repeat the experiment of doing three coin flips, in the long run, the ratio of times the experiment yielded zero, one, two or three "heads" would be roughly in the ratio of $1:3:3:1$.

Recall that in Section 7 we also interpreted the expansion of $(L+R)^n$ as listing selections of L or R from each term when they are written like this:

$$(L+R)^n = (L+R)(L+R)(L+R)\cdots(L+R).$$

But selecting an L or R is like telling whether the coin came up "heads" or "tails" in each of the terms.

There is nothing special about three flips, obviously, so if the experiment is to do n flips, then there are 2^n possible outcomes, and if all you care about is the number of times T occurred, and not on the actual order of the T and H results that generated it, then there are $\binom{n}{0}$ ways to obtain zero "heads", $\binom{n}{1}$ ways to obtain one "head", and in general, $\binom{n}{k}$ ways to obtain exactly k "heads".

In probability terms, the probability of obtaining exactly k heads in n flips is:

$$\frac{1}{2^n}\binom{n}{k}.$$

What if your experiment is not with fair coins, but rather a repeated test where the odds are the same for each test? For example, suppose the game is to roll a single die n times, and you consider it a win if a 1 occurs, but a loss if $2, 3, \ldots, 6$ occurs? Thus you win, on average, one time in six, or equivaltently, with a probability of $1/6$. If you repeat the rolling n times, what is the probability of getting exactly k wins in this situation?

We can describe any experiment like this by labeling the probability of success as p and the probability of failure as q such that $p + q = 1$ (in other words, you either win or lose – there are no other possibilities). For flipping a fair coin, $p = q = 1/2$; for the dice experiment described above, $p = 1/6$ and $q = 5/6$.

The analysis can begin as before, where we just list the possible outcomes. Using "W" for "win" and "L" for "lose", the results of three repeats are the familiar:

$$WWW, WWL, WLW, WLL, LWW, LWL, LLW, LLL.$$

But the chance of getting a W is now different from the chance of getting a L. What is the probability of getting each of the results above. For any particular set, say LWL, to obtain that, you first lose (with probability q) then you win (with probability p) and then you lose again (with probability q again). Thus the chance that that particular result occurs is qpq. For the three-repeat experiment, the chances of 0, 1, 2 and 3 wins ($P(0)$, $P(1)$, $P(2)$ and $P(3)$) are given by:

$$\begin{aligned} P(0) &= qqq = q^3 \\ P(1) &= pqq + qpq + qqp = 3pq^2 \\ P(2) &= ppq + pqp + qpp = 3p^2q \\ P(3) &= ppp = p^3 \end{aligned}$$

Notice that there's nothing special about repeating the experiment three times. If the experiment is repeated n times, the probability of obtaining exactly k wins is given by the formula:

$$P(k) = \binom{n}{k} p^k q^{n-k}.$$

Thus if you roll a fair die 7 times, the probability that you will obtain exactly 2 wins is given by:

$$P(2) = \binom{7}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^5 = \frac{21 \cdot 1 \cdot 3125}{279936} = \frac{65625}{279936} \approx 0.2344286$$

Finally, note that there's no need for the same experiment to be repeated. If you take a handful of ten coins and flip them all at once, the odds of getting, say, exactly four heads is the same as the odds of getting four heads in ten individual flips of the same coin. Just imagine flipping the first, then the second, and so on, and leaving them in order on the table after each flip.

9 Back to Combinatorics

OK, now, in principle, we can calculate any binomial coefficient simply using addition over and over to obtain the entries in the appropriate row of Pascal's triangle. If you need a number like (95 choose 11), however, this would take a *long* time, starting from scratch. The goal of this section is to show that:

$$(n \text{ choose } k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

As before, the best way to begin is with a concrete example, and we'll use (4 choose 3). One approach is to think about it this way: There are 4 ways to choose the first object, and after that is chosen, 3 ways to choose the second (since one is already picked) and finally, 2 ways to choose the last one. So there should be $4 \times 3 \times 2 = 24$ ways to do it. This, of course, conflicts with our previous result that (4 choose 3) = 4, so what's going on?

Let's again use the set $\{A, B, C, D\}$ as the set of four objects. If we make the $4 \times 3 \times 2$ choices as above, here are the sets we obtain (in alphabetical order, to be certain we've omitted nothing):

$$\begin{array}{cccccc} ABC & ABD & ACB & ACD & ADB & ADC \\ BAC & BAD & BCA & BCD & BDA & BDC \\ CAB & CAD & CBA & CBD & CDA & CDB \\ DAB & DAC & DBA & DBC & DCA & DCB \end{array}$$

The problem becomes obvious: we've included lots of groups that are identical: $ABC = ACB = BAC$ and so on. We want to count groups where the ordering doesn't matter and we've generated groups that have an order. Let's regroup the list above so that each row contains only simple rearrangements of the same items:

$$\begin{array}{cccccc} ABC & ACB & BAC & BCA & CAB & CBA \\ ABD & ADB & BAD & BDA & DAB & DBA \\ ACD & ADC & CAD & CDA & DAC & DCA \\ BCD & BDC & CBD & CDB & DBC & DCB \end{array}$$

Notice that each appears exactly 6 times, so the number 24 we obtained has counted each subset 6 times. To find the true number of subsets, we have to divide 24 by 6 and we obtain the correct answer, 4.

How many rearrangements are there of 3 items? Well, the first can be any of 3, then there remain 2 choices for the second, and the final item is determined. The result is $3 \times 2 \times 1 = 3! = 6$. Similarly, there are $4 \times 3 \times 2 \times 1 = 24$ rearrangements of 4 items and so on.

As before, there's nothing special about this method to calculate (4 choose 3). If we want to find out how many combinations there are of k things from a set of n, we say that the first can be any of n, the second any of $n - 1$, and so on, for k terms. But when we do this, we'll obtain every possible rearrangement of those k terms so we will have counted each one $k(k-1)(k-2)\cdots 3 \cdot 2 \cdot 1 = k!$ times.

Putting this together, we obtain a simple method to do the calculation. Here are a couple of examples:

$$\begin{aligned} (7 \text{ choose } 4) = \binom{7}{4} &= \frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 35 \\ (9 \text{ choose } 3) = \binom{9}{3} &= \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84 \\ (11 \text{ choose } 5) = \binom{11}{5} &= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 462 \end{aligned}$$

Notice how easy this is. If you're choosing k things from a set of n, start multiplying the numbers n, $n - 1$, and so on for k terms, and then divide by the k terms of $k!$. If we count carefully, we can see that the general formula looks like this:

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

The form above is a little inconvenient to use mathematically because of the numerator, but notice that we can convert the numerator to a pure factorial if we multiply it all the rest of the way down, which is to say, multiply the numerator by $(n-k)(n-k-1)\cdots 3\cdot 2\cdot 1 = (n-k)!$. So multiply both numerator and denominator of the equation above by $(n-k)!$ to obtain the result we wanted:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{3}$$

If you need to do an actual calculation of this sort, use the first form, since massive canceling will occur. In the example:

$$(11 \text{ choose } 5) = \binom{11}{5} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

we can cancel the 10 in the numerator by the 5 and 2 in the denominator. The 4 in the denominator cancels the 8 upstairs to a 2, and the 3 similarly cancels with the 9 yielding 3, and the problem reduces to:

$$(11 \text{ choose } 5) = \binom{11}{5} = \frac{11\cdot 3\cdot 2\cdot 7}{1} = 462.$$

The form in equation 3 is much easier to calculate with algebraically. For example, if we took this as the definition of the terms in Pascal's triangle, we could show that each row is obtained from the previous by adding the two above it if we could show that:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Just for the algebraic exercise, let's do this calculation by converting the terms to the equivalent factorial forms. We need to show that:

$$\frac{n!}{k!(n-k)!} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}.$$

To do so, all we need to do is to covert the terms on the right so that they have a common denominator and then add them together. The common denominator is $k!(n-k)!$.

$$\begin{aligned}
\frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} &= \frac{(n-1)!(n-k)}{k!(n-k)(n-k-1)!} + \frac{(n-1)!k}{k(k-1)!(n-k)!} \\
&= \frac{(n-1)!(n-k)}{k!(n-k)!} + \frac{(n-1)!k}{k!(n-k)!} \\
&= \frac{(n-k+k)(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!},
\end{aligned}$$

which is what we needed to show.

Notice also that the factorial form shows instantly that $\binom{n}{k} = \binom{n}{n-k}$; in other words, that choosing which of the k items to include gives the same value as choosing the $n-k$ items to omit.

Finally, it's probably a good idea if the students haven't seen it, to point out that these binomial coefficients can be used to find things like lottery odds. If you need to make 6 correct picks from 50 choices to win the lottery, what are the chances of winning? Well, there are $\binom{50}{6} = 15890700$ equally likely choices, so you'll win about one time in every 16 million.

10 An "Unrelated" Problem

Suppose we have a grid of city streets, with m north-south streets and n east-west streets. Figure 5 illustrates an example with $m = n = 9$ although there is no need for the two to be the same. The goal is to find the number of paths from one corner to the opposite corner (A to B in the figure) that are the shortest possible distance, in other words, with no backtracking. A typical shortest route is shown as a bold path on the grid in the figure. We will examine this problem in a couple of different ways.

One way to think of it (using the example in the figure) is that the entire route has to include 8 steps down (and 8 to the right, of course). But those 8 downward steps have to occur distributed among the 9 streets that go down. In the example route, 2 steps down are taken on the fourth street, 4 more on the fifth street, and 1 more on each of the eighth and ninth streets. If you think about it, simply knowing how many of the downward steps are taken on each of the 9 streets completely determines the route.

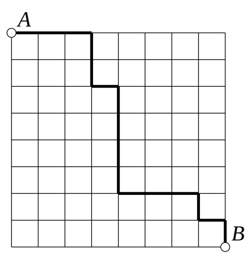

Figure 5: Routes through a grid

So the problem is equivalent to the following: How many ways are there to assign 8 identical balls (steps down) into 9 labeled boxes (the up-down streets)? This is similar to the "n choose k" type problems, but not quite the same. But here's a nice way to visualize the "identical balls in non-identical boxes" problem. Imagine that the boxes are placed side-by-side next to each other, and that we use a vertical bar to indicate the boundary between adjacent boxes. Since there are 9 boxes in this example, there will be 8 boundary walls. Similarly, let's represent the balls by stars, and there will be 8 of those.

We claim that *every* listing of vertical bars and asterisks corresponds to exactly one valid shortest-path through the grid. For example, the path in the example corresponds to this:

$$|||**|****|||*|*$$

Thus the number of paths simply corresponds to the number of arrangements of 8 bars and 8 stars. Well, we are basically writing down 16 symbols in order, and if we know which 8 of them are stars, the others are bars, so there must be (16 choose 8) = 12870 ways to do this.

If there are m streets by n streets, there are $n-1$ steps down to be distributed among m streets. Thus there will be $m-1$ vertical bars and $n-1$ asterisks. The answer is that there are ($m+n-2$ choose $n-1$) routes in an $m \times n$ grid.

It's actually probably worth counting the streets in a few simple grids before pulling out the big guns and obtaining a general solution in a classroom setting.

Now look at the same problem in a different way. Suppose we begin at point A. There is exactly one way to get there: do nothing. Now look at the points on the horizontal street from A or the vertical street from A. For every one of those points, there's only one shortest path, the straight line. What we're going to do is label all the points on the grid with the number of shortest paths there are to get there. From these simple observations, all the labels on the top and left edges of the grid will be labeled with the number 1.

1	1	1	1	1	1
1	2	3	4	5	6
1	3	6	10	15	21
1	4	10	20	35	56
1	5	15	35	70	126
1	6	21	56	126	252

Figure 6: Counting paths in a grid

Now for the key observation: to get to any point inside the grid, you either arrived from above or from the left. The total number of unique routes to that point will be the total number of routes to the point above plus the total number of routes to the point on your left. The upper left corner of the resulting grid will thus look something like what is illustrated in figure 6.

Notice that this is just Pascal's triangle, turned on its side!

In fact, if we start at the apex of Pascal's triangle and take paths that always go down, but can go either to the right or left at each stage, then the numbers in the triangle indicate the number of paths by which they can be reached. In the early examples with the binomial theorem, we used R and L, that we can now think of as "right" and "left". When we reach the position in Pascal's triangle corresponding to (7 choose 3) what that really amounts to is the number of paths from the apex that are 7 steps long, and which contain 3 moves to the left (and hence $7 - 3 = 4$ moves to the right).

11 Binomial Coefficient Relationships

Here is a series of identities satisfied by the binomial coefficients. Some are easy to prove, and some are difficult.

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n \tag{a}$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n} = 0, n > 0 \tag{b}$$

$$\frac{1}{1}\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1} \tag{c}$$

$$0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1} \tag{d}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \binom{n}{3}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n} \tag{e}$$

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \binom{n}{3}^2 + \cdots + (-1)^n\binom{n}{n}^2 = \begin{cases} 0 & : \quad n = 2m+1 \\ (-1)^m\binom{2m}{m} & : \quad n = 2m \end{cases} \tag{f}$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \binom{n}{8} + \binom{n}{10} + \cdots = 2^{n-1}, n > 0 \tag{g}$$

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \binom{n}{7} + \binom{n}{9} + \binom{n}{11} + \cdots = 2^{n-1}, n > 0 \tag{h}$$

Relations (a) and (b) were proved earlier in this article (see Section 3).

Relations (c) and (d) can be proved by standard methods, but a quick proof is available if we use a trick from calculus. We begin by noticing that the binomial theorem tells us that:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

If we take the derivative of both sides with respect to x, we obtain:

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 \cdots + n\binom{n}{n}x^{n-1}.$$

Substitute 1 for x and we obtain relation (d).

If we start from the same equation but integrate instead, we obtain:

$$\frac{(1+x)^{n+1}}{n+1} = \binom{n}{0}x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \cdots + \frac{1}{n+1}\binom{n}{n}x^{n+1} + C,$$

where C is a constant to be determined. Substituting $x = 0$, we obtain $C = 1/(n+1)$, and then if we substitute $x = 1$, we obtain relation (c).

If you don't want to use calculus, here's a different approach. For relation (d) note that:

$$k\binom{n}{k} = \frac{kn!}{(n-k)!k!} = \frac{n(n-1)!}{(n-k)!(k-1)!} = n\binom{n-1}{k-1}.$$

Then

$$\begin{aligned} 0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} &= \\ n\left(\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1}\right) &= n2^{n-1}. \end{aligned}$$

Relation (c) can be proved similarly, starting from the fact that:

$$\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k+1}.$$

We'll leave the proof as an exercise.

There are different ways to prove relation (e), but my favorite is a combinatorial argument. Since $\binom{n}{k} = \binom{n}{n-k}$ we can rewrite the sum of the squares of the binomial coefficients as:

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots \binom{n}{n}\binom{n}{0}.$$

Now let's consider a particular way to calculate $\binom{2n}{n}$. This just counts the number of ways to choose n items from a set of $2n$. Imagine that we divide the $2n$ items into two sets, A and B, each of which contains n items. If we choose none of the items in set A, we have to choose all n from set B. If we choose 1 item from A we have to choose $n-1$ from B, and so on. In general, for each of the $\binom{n}{k}$ ways we can choose k items from set A, there are $\binom{n}{n-k}$ ways to choose the remaining items from B, and this will add $\binom{n}{k}\binom{n}{n-k}$ more ways to choose n items from $2n$. Sum them up, and we obtain relation (e).

A more traditional way to do the calculation is as follows. First we note that:

$$\begin{aligned} (x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n \\ &= \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1}y + \binom{n}{n-2}x^{n-2}y^2 + \cdots + \binom{n}{0}y^n \end{aligned}$$

If we multiply $(x+y)^n$ by itself, once represented by the first version above and once represented by the second, we can see that the term containing x^ny^n will have a coefficient equal to the desired sum of products of binomial coefficients. But this is just the same as the coefficient of x^ny^n in the expansion of $(x+y)^n(x+y)^n = (x+y)^{2n}$, and that is just $\binom{2n}{n}$.

Finally, a third way to see that the sum is correct is to consider a special case of counting routes through a grid that we solved in Section 10.

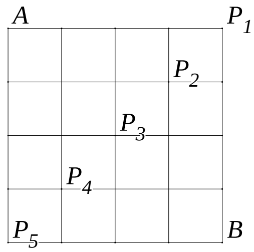

Figure 7: Routes in a square grid

Consider a square grid such as the one displayed in Figure 7. In that figure there are five horizontal and five vertical paths, but the argument we will make will not depend on that. Suppose we want to count the number of paths from point A to point B in that figure. From the work we did in Section 10, we know that the number is $\binom{8}{4}$.

But any path from A to B must pass through exactly one of the points P_1, P_2, $\ldots$, P_5. The number of paths from A to B passing through point P_i is the number of paths from A to P_i multiplied by the number of paths from P_i to B.

Again, using methods from Section 10, there are $\binom{4}{0}$ paths from A to P_1 and $\binom{4}{0}$ paths from P_1 to B. There are $\binom{4}{1}$ paths from A to P_2 and $\binom{4}{1}$ paths from P_2 to B, and so on. Thus the total number of paths, which we know to be $\binom{8}{4}$, is:

$$\binom{4}{0}\binom{4}{0}+\binom{4}{1}\binom{4}{1}+\binom{4}{2}\binom{4}{2}+\binom{4}{3}\binom{4}{3}+\binom{4}{4}\binom{4}{4}=\binom{8}{4}.$$

It should be clear that there is nothing special about a 5×5 grid, and that the same argument can be used to count paths through *any* square grid, and this yields the result expressed in relation (e).

The second method that we used to compute relation (e) can be used to prove relation (f), and all we have to do is to multiply $(x-y)^n$ by $(x+y)^n$ using the following two expansions:

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n}y^n\\(x-y)^n &= \binom{n}{n}x^n-\binom{n}{n-1}x^{n-1}y+\binom{n}{n-2}x^{n-2}y^2-\cdots+(-1)^n\binom{n}{0}y^n\end{aligned}$$

Again, the desired term will be the coefficient of x^ny^n in the expansion of $(x+y)^n(x-y)^n$, but this is just $(x^2-y^2)^n$. If n is odd, this coefficient is clearly zero, and if n is even, say $n=2m$, then it is $(-1)^m\binom{2m}{m}$.

The case where n is odd is totally obvious since the values in Pascal's triangle are symmetric, and there is no middle term for odd n, so each term with a positive sign will be matched by the corresponding symmetric term with a negative sign.

Relations (g) and (h) are easy to obtain. Consider the following two expansions:

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n}y^n\\(x-y)^n &= \binom{n}{0}x^n-\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2-\cdots+(-1)^n\binom{n}{n}y^n\end{aligned}$$

If we add the equations together, we obtain twice the sum in relation (g) and if we subtract them, we obtain twice the sum in (h). Then simply set x and y to 1, and we see that the sum *or* difference is 2^n. Since this is twice the desired result, we see that in both cases, the sum is 2^{n-1}.

12 Harmonic Differences

Write down the fractions in the harmonic series: $1/1$, $1/2$, $1/3$, $\ldots$, and in each row below that, calculate the differences of the adjacent numbers. You will obtain a table that looks something like this:

$$\begin{array}{ccccccccccccc}
\frac{1}{1} & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \frac{1}{6} & & \cdots \\
& \frac{1}{2} & & \frac{1}{6} & & \frac{1}{12} & & \frac{1}{20} & & \frac{1}{30} & & \cdots & \\
& & \frac{1}{3} & & \frac{1}{12} & & \frac{1}{30} & & \frac{1}{60} & & \cdots & & \\
& & & \frac{1}{4} & & \frac{1}{20} & & \frac{1}{60} & & \cdots & & & \\
& & & & \frac{1}{5} & & \frac{1}{30} & & \cdots & & & & \\
& & & & & \frac{1}{6} & & \cdots & & & & &
\end{array}$$

If this table is rotated until the fraction $1/1$ is on top, and then if the top row is multiplied by 1, the second row by 2, the third by 3 and so on, we obtain a triangle that is exactly the same as Pascal's triangle except that the numbers are in the denomintors instead of the numerators. This is easy to prove if we simply write down the fractions in the table above in terms of the binomial coefficients and show that they satisfy the difference equations. Following is the table above written in that form.

$$\begin{array}{ccccccccccccc}
\left(1\binom{0}{0}\right)^{-1} & & \left(2\binom{1}{1}\right)^{-1} & & \left(3\binom{2}{2}\right)^{-1} & & \left(4\binom{3}{3}\right)^{-1} & & \left(5\binom{4}{4}\right)^{-1} & & \left(6\binom{5}{5}\right)^{-1} & \cdots & \\
& \left(2\binom{1}{0}\right)^{-1} & & \left(3\binom{2}{1}\right)^{-1} & & \left(4\binom{3}{2}\right)^{-1} & & \left(5\binom{4}{3}\right)^{-1} & & \left(6\binom{5}{4}\right)^{-1} & & \cdots & \\
& & \left(3\binom{2}{0}\right)^{-1} & & \left(4\binom{3}{1}\right)^{-1} & & \left(5\binom{4}{2}\right)^{-1} & & \left(6\binom{5}{3}\right)^{-1} & & \cdots & & \\
& & & \left(4\binom{3}{0}\right)^{-1} & & \left(5\binom{4}{1}\right)^{-1} & & \left(6\binom{5}{2}\right)^{-1} & & \cdots & & & \\
& & & & \left(5\binom{4}{0}\right)^{-1} & & \left(6\binom{5}{1}\right)^{-1} & & \cdots & & & & \\
& & & & & \left(6\binom{5}{0}\right)^{-1} & & \cdots & & & & &
\end{array}$$

It is obvious that the first row is just the harmonic series, so all that needs to be shown is that successive entries in the table can be obtained by subtracting the two entries above. This is equivalent to showing that for any n and k:

$$\frac{1}{(n+1)\binom{n}{k}} - \frac{1}{(n+2)\binom{n+1}{k+1}} = \frac{1}{(n+2)\binom{n+1}{k}}.$$

If we write the binomial coefficients above in their factorial form, this is equivalent to:

$$\frac{(n+2)k!(n-k)!}{n!} - \frac{(n+1)(k+1)!(n-k)!}{(n+1)!} = \frac{(n+1)k!(n+1-k)!}{(n+1)!}.$$

We can put everything over a common denominator of $n!$:

$$\frac{(n+2)k!(n-k)!}{n!} - \frac{(k+1)!(n-k)!}{n!} = \frac{k!(n+1-k)!}{n!},$$

so since the denominators are the same, we are done if:

$$(n+2)k!(n-k)! - (k+1)!(n-k)! \quad = \quad k!(n+1-k)!$$

$$
\begin{aligned}
k!((n+2)(n-k)! - (k+1)(n-k)!) &= k!(n+1-k)! \\
k!(n+2-k-1)(n-k)! &= k!(n+1-k)! \\
k!(n-k+1)(n-k)! &= k!(n+1-k)! \\
k!(n+1-k)! &= k!(n+1-k)!
\end{aligned}
$$

13 Finding Formulas for Sequences

Suppose you come across the following sequence:

$$5, 7, 21, 53, 109, 195, 317, 481, \ldots,$$

and you would like to find a general formula $f(n)$ such that $f(0) = 5$, $f(1) = 7$, $f(2) = 21$, $f(3) = 53$, and so on. In many, many cases the following technique works.

$$
\begin{array}{ccccccccccccccccc}
5 & & 7 & & 21 & & 53 & & 109 & & 195 & & 317 & & 481 & & \ldots \\
& 2 & & 14 & & 32 & & 56 & & 86 & & 122 & & 164 & & \ldots & \\
& & 12 & & 18 & & 24 & & 30 & & 36 & & 42 & & \ldots & & \\
& & & 6 & & 6 & & 6 & & 6 & & 6 & & \ldots & & & \\
& & & & 0 & & 0 & & 0 & & 0 & & \ldots & & & &
\end{array}
$$

First list the numbers in a line as in the table above, and on each successive line, write the difference of the pair of numbers above it. This is sort of the opposite of what you do to form Pascal's triangle. Continue in this way, and if you are lucky, you will wind up with a line that remains constant (the line of 6's above), and then, if you were to continue, the next line and every other line would be completely filled with zeros. (If this does not happen, then the following technique will not work, although if you do find a pattern, other techniques may work.)

The example above happened to degenerate to all zeroes in the fifth line. This may occur in more or fewer lines. The sequence $1, 3, 5, 7, 9, \ldots$ is all zeros on the third line, and other sequences may require more lines before they degenerate to all zeroes.

This isn't exactly a formula, but it does provide an easy method to compute successive values in the original list, assuming the pattern continues. In this case, since the line of 6's continues, just add another 6 to it. That means that the number following the 42 in the line above must be $42 + 6 = 48$. The same reasoning tells us that the number following the 164 must be $164 + 48 = 212$ and repeating the reasoning, the next number in our sequence will be $481 + 212 = 693$. We can repeat this as often as we like to obtain an arbitrary number of terms.

The most important thing to note is that the entire table is completely determined by the numbers on the left edge; in this case they are: $5, 2, 12$ and 6 followed by an infinite sequence of zeroes.

Although the method above will certainly work, it would still be painful to calculate the millionth term this way. It would be much better to have an explicit formula for the n^{th} term.

As we already observed, the generation of tables like those above is similar to what we do to obtain Pascal's triangle except that we subtract instead of add. Because of this it is not surprising that an explicit formula can be obtained based on coefficients in Pascal's triangle. We will simply state the method for obtaining such a formula and then we will explore why it works. In fact, our formula will use only the numbers on the left edge of our table (which, as we observed, generate the entire table). In the example, they are $5, 2, 12, 6, 0, 0, \ldots$. Here is the formula:

$$5\binom{n}{0} + 2\binom{n}{1} + 12\binom{n}{2} + 6\binom{n}{3} + 0\binom{n}{4} + 0\binom{n}{5} + \cdots$$

Obviously once we arrive at the infinite sequence of zeros, none of the remaining terms will contribute anything, so the formula above really only has four non-zero terms. In its current state the formula is correct, but it can be simplified with a bit of algebra:

$$\begin{aligned}
f(n) &= 5\binom{n}{0} + 2\binom{n}{1} + 12\binom{n}{2} + 6\binom{n}{3} \\
f(n) &= 5 \cdot 1 + 2 \cdot n + 12 \cdot \frac{n(n-1)}{2} + 6 \cdot \frac{n(n-1)(n-2)}{6} \\
f(n) &= 5 + 2n + 6(n^2 - n) + (n^3 - 3n^2 + 2n) \\
f(n) &= n^3 + 3n^2 - 2n + 5.
\end{aligned}$$

You can check that this formula is correct by plugging in different values for n and verifying that $f(0) = 5$, $f(1) = 7$, $f(2) = 21$, and so on.

The method seems to work great, but why? One way to see this is to apply the method to a diagonal of Pascal's triangle where $f(x) = \binom{n}{3}$. (There is nothing special about this row except that it is the most complex part of the formula we generated for the example above.) We will make the sensible assumptions that $\binom{0}{3} = \binom{1}{3} = \binom{2}{3} = 0$. In other words there is no way to choose 3 objects from a set consisting of zero, one or two objects. Here is a table of successive differences with $\binom{n}{3}$ as the top row:

$$\begin{array}{ccccccccccccccccc}
0 & & 0 & & 0 & & 1 & & 4 & & 10 & & 20 & & 35 & & \ldots \\
& 0 & & 0 & & 1 & & 3 & & 6 & & 10 & & 15 & & \ldots & \\
& & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & \ldots & & \\
& & & 1 & & 1 & & 1 & & 1 & & 1 & & \ldots & & & \\
& & & & 0 & & 0 & & 0 & & 0 & & \ldots & & & &
\end{array}$$

What jumps out is that this is just a part of Pascal's triangle on its side. Each successive row in the table is another diagonal in Pascal's triangle. If you look at how the numbers are generated, it is obvious why this occurs, and that no matter what diagonal of the triangle you begin with, a similar pattern will occur.

Note also that the "generating pattern" of numbers on the left of the table is, in this case, $0, 0, 0, 1, 0, 0, \ldots$. Thus $0, 0, 0, 1, 0, 0, \ldots$ generates the row $\binom{n}{3}$. If we had begun with the row $\binom{n}{2}$, the corresponding generating pattern would have been $0, 0, 1, 0, 0, \ldots$, or for $\binom{n}{5}$, it would have been $0, 0, 0, 0, 0, 1, 0, 0, \ldots$ and so on.

If the generating pattern for some sequence were $0, 1, 0, 0, 1, 0, 0, \ldots$, it should be clear that the initial line will look like:

$$0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 0 \cdot \binom{n}{2} + 0 \cdot \binom{n}{3} + 1 \cdot \binom{n}{4}.$$

Also note that if we had multiplied every term in the top row by a constant, then every row of successive differences would be multiplied by the same constant, and the final pattern will simply have that constant in the approprite spots in the list of numbers on the left of the table. Thus our technique of multiplied the appropriate coefficient of Pascal's triangle by the constant in that row and summing them should yield a formula for the sequence.

As a final example, let's work out a formula for the following sum:

$$f(n) = 0^3 + 1^3 + 2^3 + 3^3 + \cdots + n^3.$$

We can work out the first few values by hand, and using our method, obtain the following table. (Note the interesting fact that all the numbers in the first row happen to be perfect squares.)

$$\begin{array}{ccccccccccccccccc}
0 & & 1 & & 9 & & 36 & & 100 & & 225 & & 441 & & 784 & & \ldots \\
& 1 & & 8 & & 27 & & 64 & & 125 & & 216 & & 343 & & \ldots & \\
& & 7 & & 19 & & 37 & & 61 & & 91 & & 127 & & \ldots & & \\
& & & 12 & & 18 & & 24 & & 30 & & 36 & & \ldots & & & \\
& & & & 6 & & 6 & & 6 & & 6 & & \ldots & & & & \\
& & & & & 0 & & 0 & & 0 & & \ldots & & & & &
\end{array}$$

According to our analysis, the result should be:

$$f(n) = 0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 7 \cdot \binom{n}{2} + 12 \cdot \binom{n}{3} + 6 \cdot \binom{n}{4}.$$

If we apply some algebra, we obtain:

$$\begin{aligned}
f(n) &= 0\binom{n}{0} + 1\binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4} \\
f(n) &= 0 \cdot 1 + 1 \cdot n + 7 \cdot \frac{n(n-1)}{2} + 12 \cdot \frac{n(n-1)(n-2)}{6} + 6 \cdot \frac{n(n-1)(n-2)(n-3)}{24} \\
f(n) &= 0 + n + 7\frac{(n^2-n)}{2} + 2(n^3 - 3n^2 + 2n) + \frac{n^4 - 6n^3 + 11n^2 - 6n}{4} \\
f(n) &= \frac{4n}{4} + \frac{14n^2 - 14n}{4} + \frac{8n^3 - 24n^2 + 16n}{4} + \frac{n^4 - 6n^3 + 11n^2 - 6n}{4} \\
f(n) &= \frac{n^4 - 2n^3 + n^2}{4} = \left(\frac{n(n+1)}{2}\right)^2.
\end{aligned}$$

The result above may be more familiar as:

$$0^3 + 1^3 + 2^3 + \cdots n^3 = (0 + 1 + 2 + \cdots + n)^2$$

since $1 + 2 + 3 + \cdots n = n(n+1)/2$.

14 An Advanced Example

Warning: this example depends on a knowledge of both trigonometry and complex numbers, including Euler's famous formula: $e^{i\theta} = \cos\theta + i\sin\theta$. Some steps will be motivated, but not proved. The ideas, however, are quite interesting, and may be fascinating to advanced high school students. A justification (that depends on calculus) for Euler's theorem can be found in Appendix A.

From trigonometry, it is not hard to derive the following formula:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

(The formula above can be derived from the well-known formulas for the sine and cosine of sums of angles.)

Using this formula repeatedly, it is not hard to find formulas in terms of $\tan\alpha$ for expressions of the form: $\tan(2\alpha)$, $\tan(3\alpha)$, $\tan(4\alpha)$, et cetera. Here are the first few examples, and to make the relationship to the binomial coefficients obvious, in the table below we have written t in place of $\tan\alpha$:

$$\begin{aligned}
\tan(2\alpha) &= \frac{2t}{1-t^2} \\
\tan(3\alpha) &= \frac{3t-t^3}{1-3t^2} \\
\tan(4\alpha) &= \frac{4t-4t^3}{1-6t^2+t^4} \\
\tan(5\alpha) &= \frac{5t-10t^3+t^5}{1-10t^2+5t^4}.
\end{aligned}$$

From the examples above, it is fairly easy to see that all the terms in the fractions are obtained by using the terms from the expansion of $(1+t)^n$ alternately in the denominator and numerator, but alternating signs in both the numerator and denominator. In other words, the signs of the first two terms are positive, of the next two are negative, and so on.

The relationship above can be demonstrated based on Euler's famous formula:

$$e^{i\alpha} = \cos\alpha + i\sin\alpha,$$

where i is the imaginary square-root of -1: $i^2 = -1$.

We will illustrate how the formula for $\tan(n\alpha)$ can be derived using the value $n = 4$. The same sort of calculation can be done for an arbitrary value of n. What we will do is to expand $e^{4i\alpha}$ in two different ways:

$$\begin{aligned}
e^{4i\alpha} &= \cos(4\alpha) + i\sin(4\alpha) \\
e^{4i\alpha} &= (e^{i\alpha})^4 = (cos\alpha + i\sin\alpha)^4.
\end{aligned}$$

The right-hand side of the second equation can be expanded using the binomial theorem in terms of the binomial coefficients as follows:

$$\begin{aligned}
(\cos\alpha + i\sin\alpha)^4 &= \cos^4\alpha + 4i\cos^3\alpha\sin\alpha + 6i^2\cos^2\alpha\sin^2\alpha + 4i^3\cos\alpha\sin^3\alpha + i^4\sin^4\alpha \\
&= \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha + i(4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha)
\end{aligned}$$

Note that in the second line, the expression has been divided into a real and imaginary part, and we have used the fact that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. If we were expanding higher powers of the expression, we would have to use facts like $i^5 = i$, $i^6 = -1$, $i^7 = -i$, and so on, where i^n cycles repeatedly through the values $i, -1, -i, 1$ forever.

Euler's formula also tells us that

$$e^{4i\alpha} = cos(4\alpha) + i\sin(4\alpha).$$

Since the cosine and sine functions output real values, we can equate the real and imaginary parts of the expansion done in two ways. Basically, we will have:

$$\begin{aligned}\cos(4\alpha) &= \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha \\ \sin(4\alpha) &= 4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha.\end{aligned}$$

Now, since $\tan(4\alpha) = \sin(4\alpha)/\cos(4\alpha)$, we have:

$$\tan(4\alpha) = \frac{\sin(4\alpha)}{\cos(4\alpha)} = \frac{4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha}{\cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha}.$$

If we now divide both the numerator and denominator of the result on the right by $\cos^4\alpha$, we obtain:

$$\begin{aligned}\tan(4\alpha) &= \frac{\left(\frac{4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha}{\cos^4\alpha}\right)}{\left(\frac{\cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + 4\sin^4\alpha}{\cos^4\alpha}\right)} \\ &= \frac{\left(4\frac{\sin\alpha}{\cos\alpha} - 4\frac{\sin^3\alpha}{\cos^3\alpha}\right)}{\left(1 - 6\frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\sin^4\alpha}{\cos^4\alpha}\right)} \\ &= \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha.}\end{aligned}$$

If you examine how the coefficients in the final expression above come from the binomial coefficients and how the powers of the imaginary number i cause the signs to alternate as they do, it is easy to see how the general formula for $\tan(n\alpha)$ in terms of $\tan\alpha$ can be obtained.

A Motivation for Euler's Theorem

This appendix depends on a knowledge of calculus, but provides a good justification (but not exactly a proof) for why Euler's Theorem:

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

might be true. The demonstration depends on Taylor's theorem for a function expanded about zero which states that for certain analytic functions $f(x)$ that:

$$f(x) = \frac{f(0)}{0!} + \frac{xf'(0)}{1!} + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{\mathrm{iv}}(0)}{4!} + \cdots \quad (4)$$

for values of x sufficiently close to zero. The functions e^x, $\cos x$ and $\sin x$ are all functions of this type, and in fact, the series above converges properly for *all* values of x for all three of those functions.

It is easy to expand the functions e^x, $\cos x$ and $\sin x$ as Taylor series, since every derivative of e^x is again e^x, and the derivatives of $\cos x$ and $\sin x$ cycle through values with a cycle length of 4:

$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{\mathrm{iv}}(x)$	$f^{\mathrm{v}}(x)$	$f^{\mathrm{vi}}(x)$	$f^{\mathrm{vii}}(x)$	$f^{\mathrm{viii}}(x)$	$f^{\mathrm{ix}}(x)$
e^x	e^x	e^x	e^x	e^x	e^x	e^x	e^x	e^x	e^x
$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$	$-\sin x$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$

Now, we can use formula 4 to find expansions of the three functions since $e^0 = 1$, $\cos 0 = 1$ and $\sin 0 = 0$ to obtain:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \end{aligned}$$

Now if we simply plug in the value $i\alpha$ for x in the formula for e^x above, we obtain:

$$\begin{aligned} e^{i\alpha} &= 1 + i\alpha - \frac{\alpha^2}{2!} - i\frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + i\frac{\alpha^5}{5!} - \frac{\alpha^6}{6!} - i\frac{\alpha^7}{7!} + \frac{\alpha^8}{8!} - \cdots \\ &= \left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots\right) + i\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \cdots\right) \end{aligned}$$

In the second line above, we have simply separated the real and imaginary parts of the expansion, and it is easy to see that the real part is the same as the Taylor expansion of $\cos\alpha$ and that the imaginary part is the Taylor expansion of $\sin\alpha$. Thus we conclude Euler's theorem:

$$e^{i\alpha} = \cos\alpha + i\sin\alpha.$$

B Pascal's Triangle

Here is a larger version of Pascal's triangle, through line 18:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1

1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1

1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

1 17 136 680 2380 6188 12376 19448 24310 24310 19448 12376 6188 2380 680 136 17 1

1 18 153 816 3060 8568 18564 31824 43758 48620 43758 31824 18564 8568 3060 816 153 18 1

C Modulo Triangles

In this section, we display Pascal's triangle as a set of green and red circles. A red circle indicates that the entry is equal to zero, modulo some number.

Notice how the figures mod primes are different from the non-primes. Can you think of why?

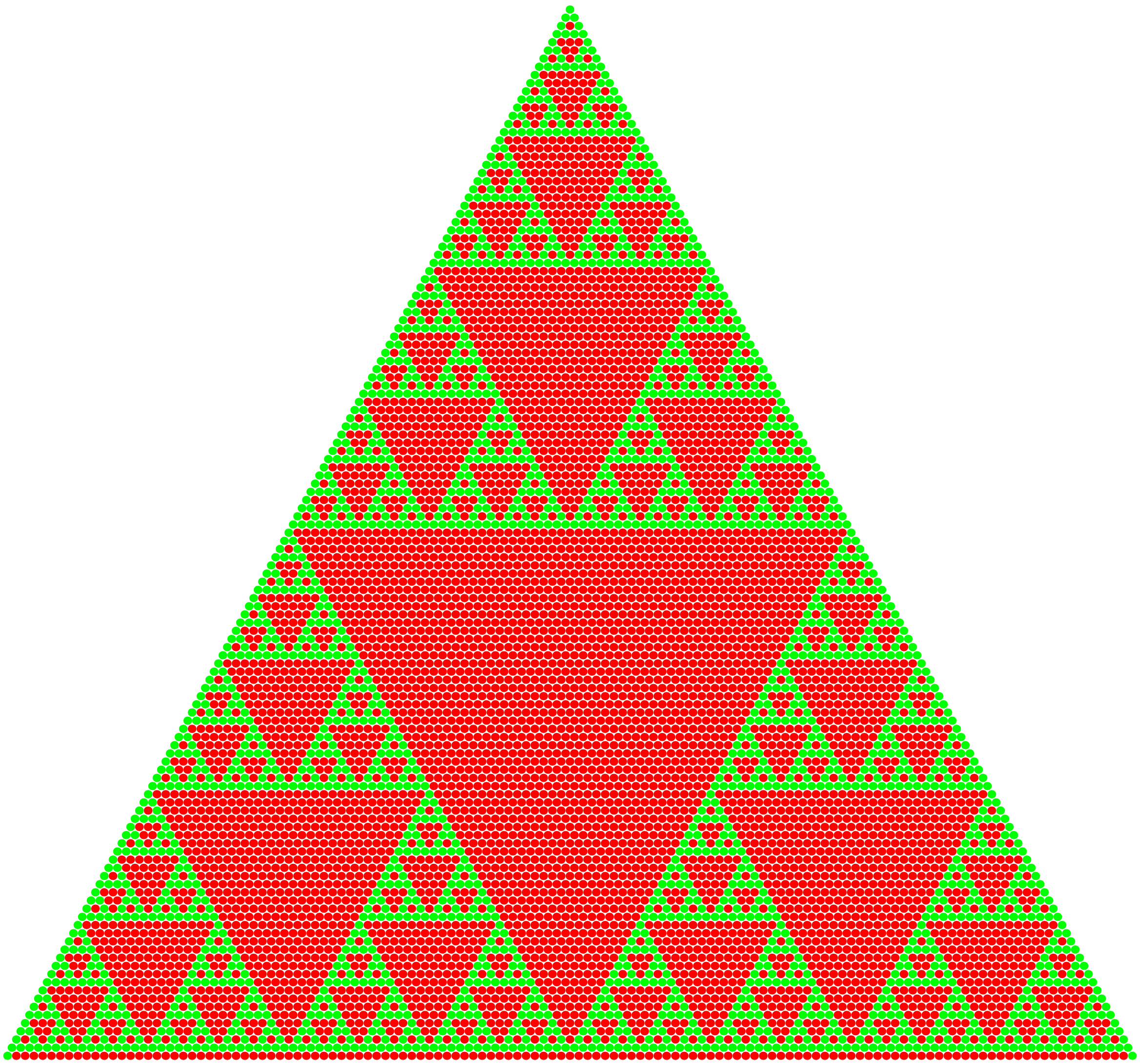

Figure 8: Pascal's Triangle, Modulo 2

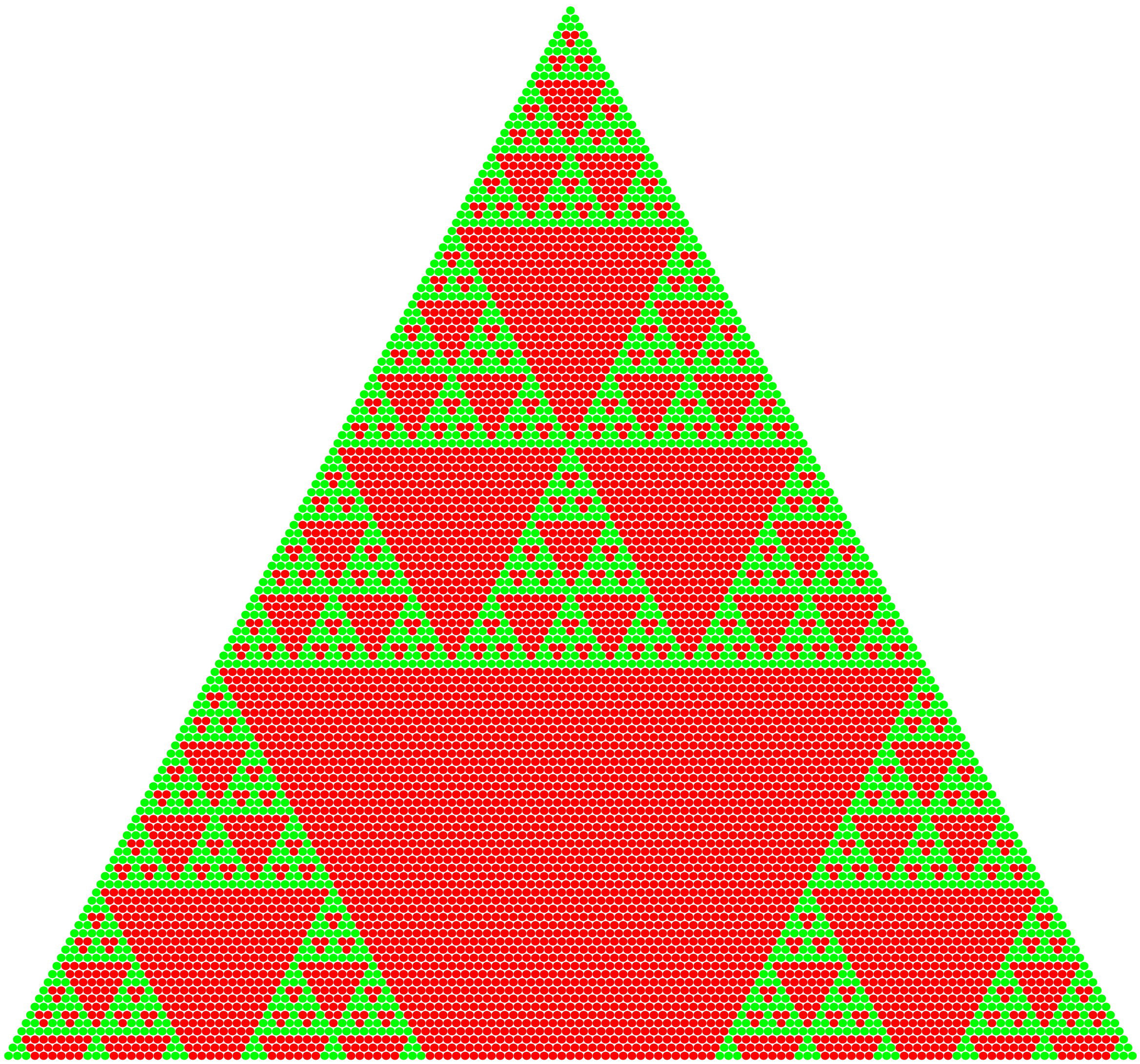

Figure 9: Pascal's Triangle, Modulo 3

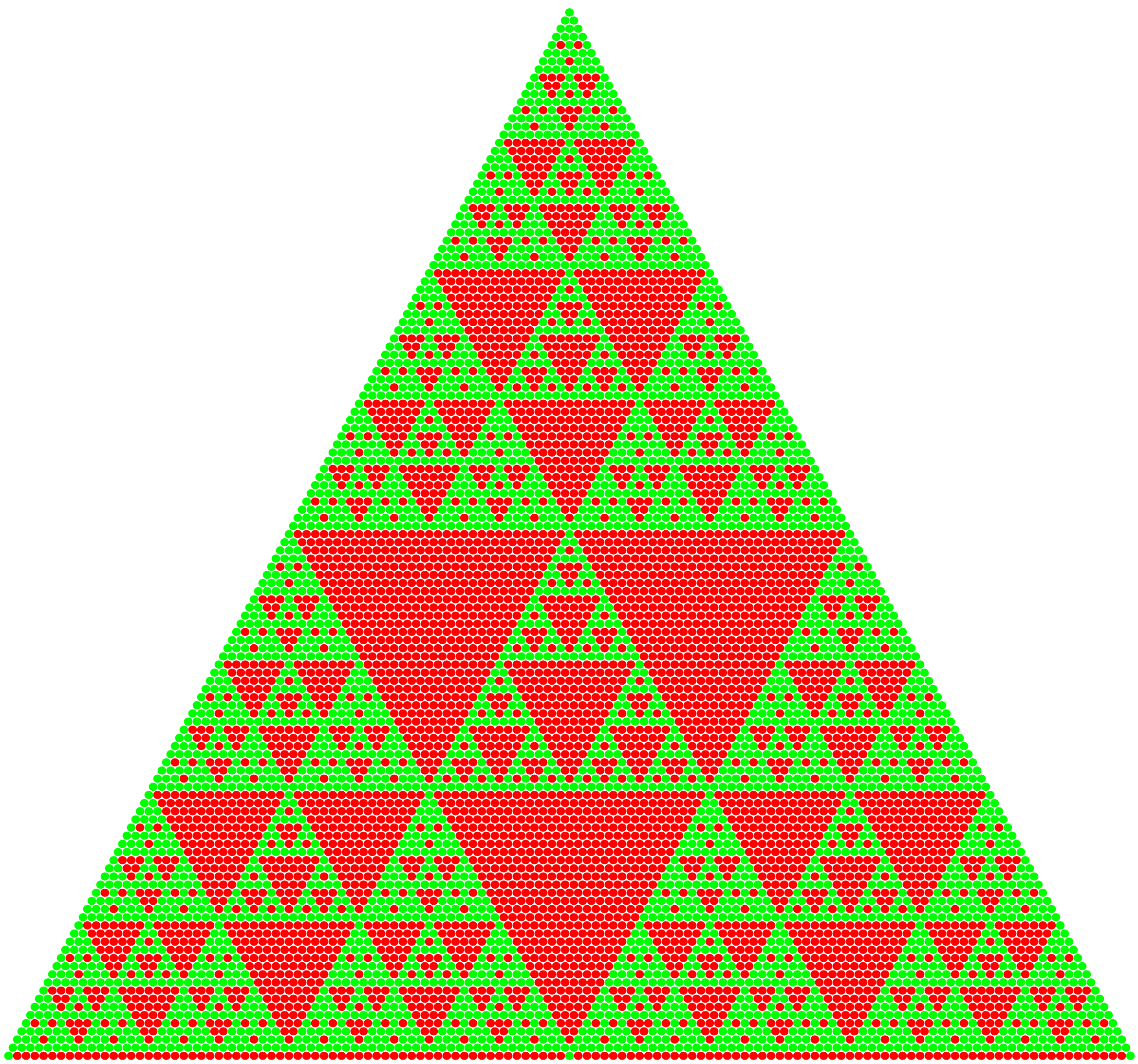

Figure 10: Pascal's Triangle, Modulo 4

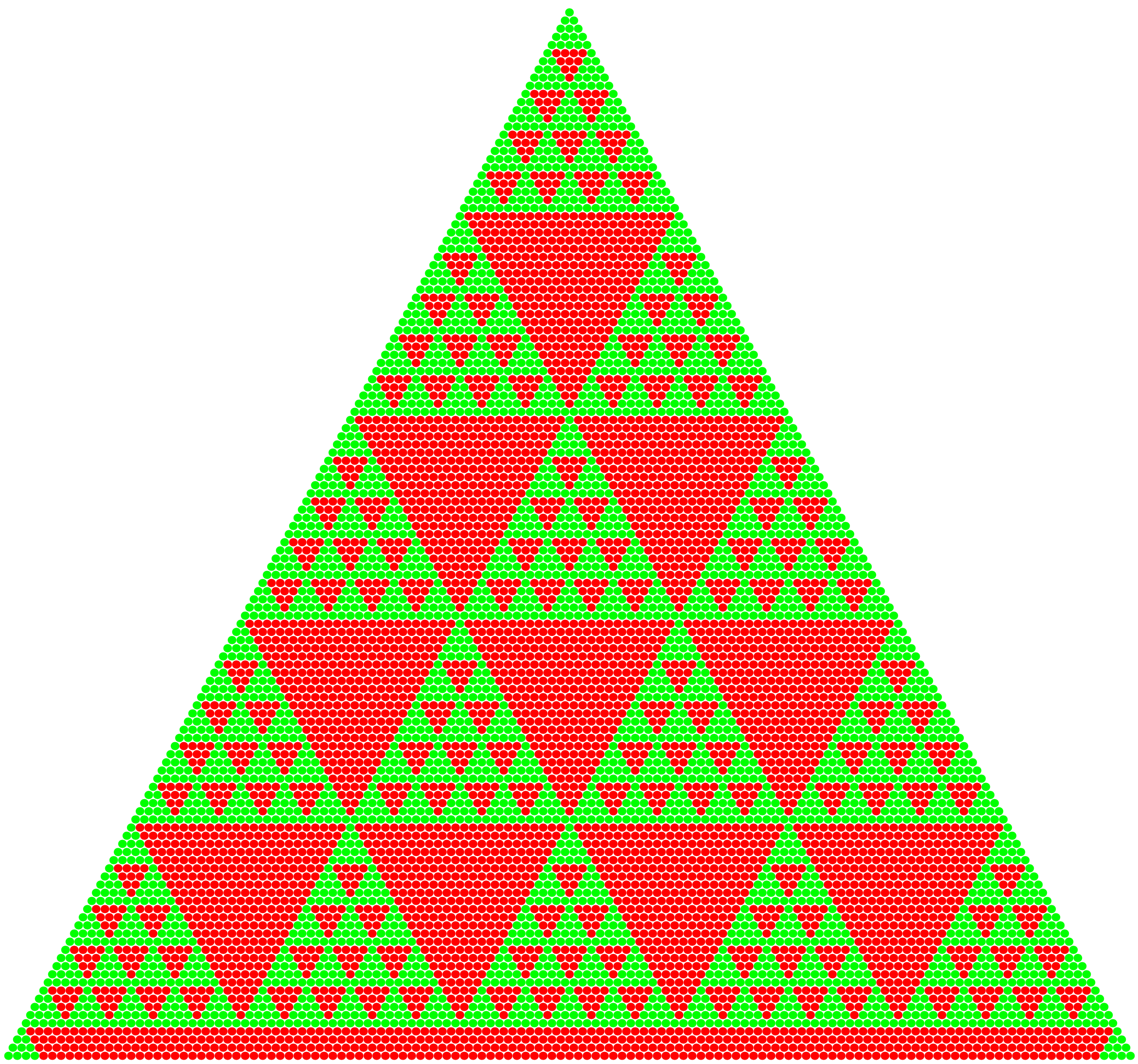

Figure 11: Pascal's Triangle, Modulo 5

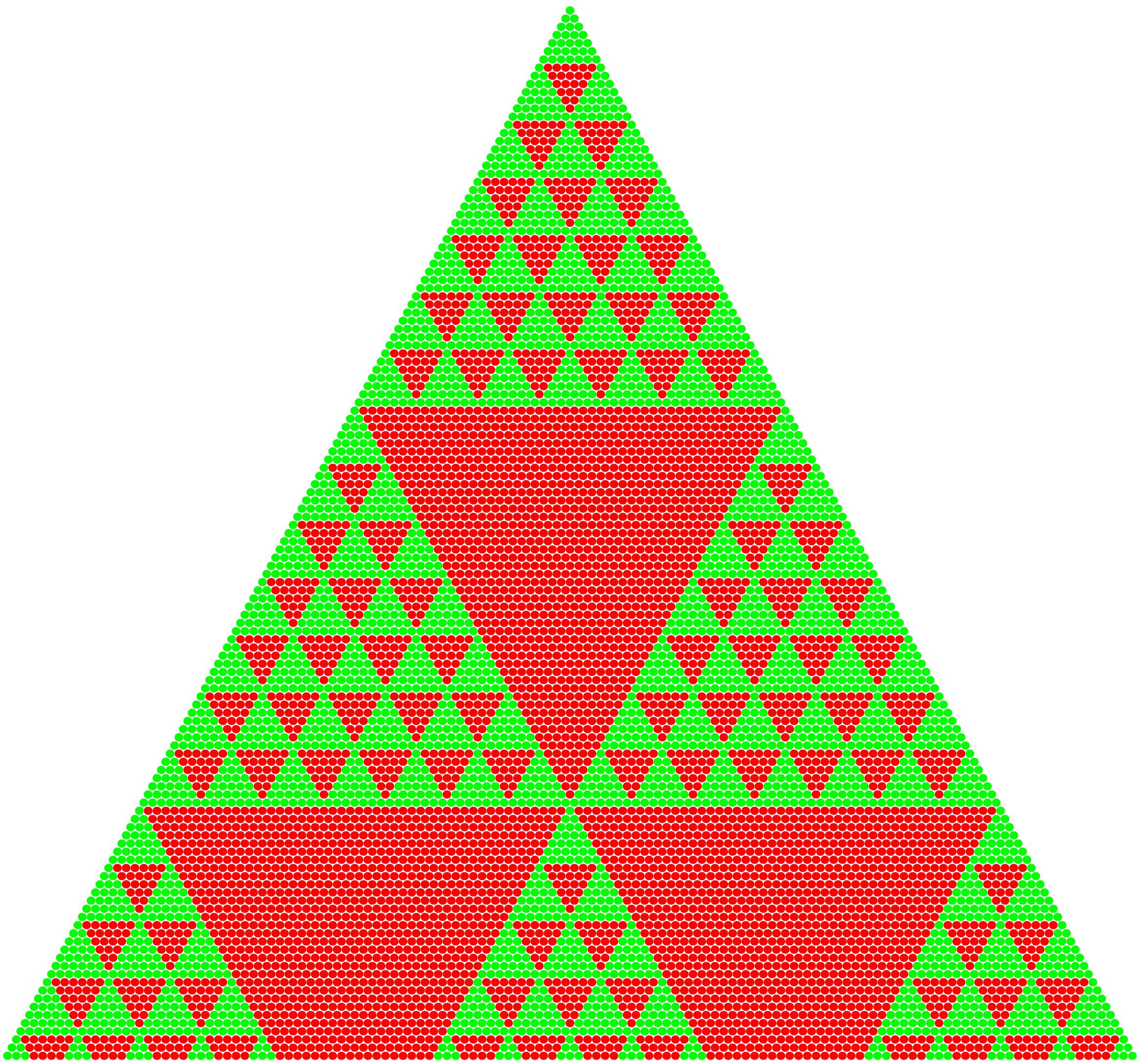

Figure 12: Pascal's Triangle, Modulo 7

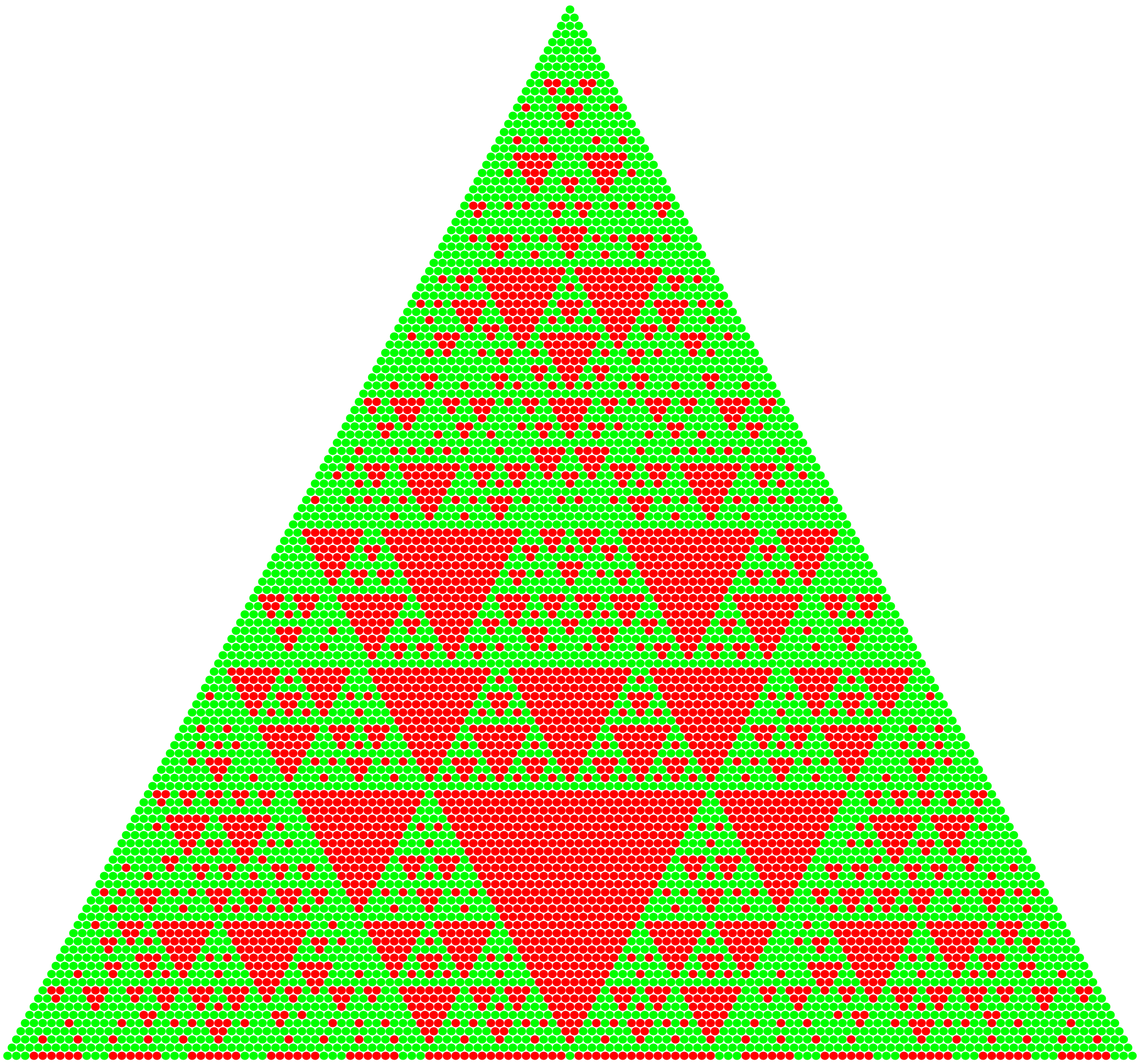

Figure 13: Pascal's Triangle, Modulo 12

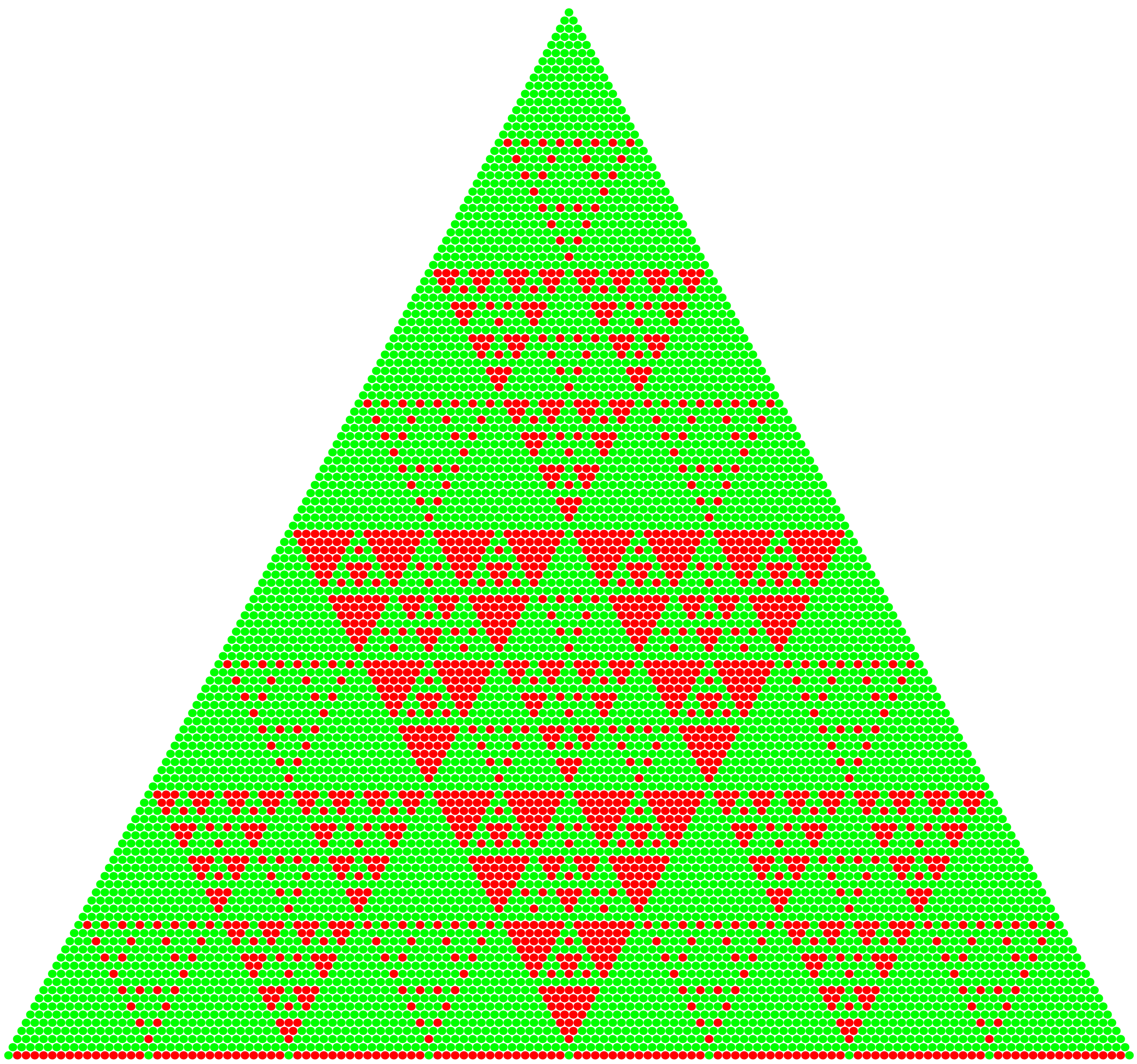

Figure 14: Pascal's Triangle, Modulo 16