Organization in Mathematics

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1 Introduction

When faced with a difficult mathematical problem, one good strategy is to thoroughly investigate the easiest cases of the problem, or to solve a simplified version of that problem. By "problem" we just mean something where you don't initially know how to go about solving it. A problem for a child may be an easy exercise for an adult, so the meaning of "problem" changes with time.

In your investigation of easy versions of the problem, you can often see possible patterns in the solutions that make it possible to guess a good approach to the more general version.

Some problems are really multiple problems with some easy and some difficult versions. For example, if you are asked to find the sum of the first n integers: find S such that:

$$S = 1 + 2 + 3 + \cdots + n,$$

in some sense we really have an infinite number of questions, or cases to consider.

Some cases are trivial: with just a little arithmetic we can solve the problem above for small values of n:

$$\begin{aligned}
n = 1 : 1 &= 1 \\
n = 2 : 3 &= 1 + 2 \\
n = 3 : 6 &= 1 + 2 + 3 \\
n = 4 : 10 &= 1 + 2 + 3 + 4 \\
n = 5 : 15 &= 1 + 2 + 3 + 4 + 5
\end{aligned}$$

Now, by looking at the sequence of results for the first 5 versions of the problem, perhaps we can see a pattern in the list: $1, 3, 6, 10, 15$. Someone who has never seen this problem before might recognize this sequence as the triangular numbers or as a diagonal in Pascal's triangle. In either case, there are formulas for these numbers and knowing a possible formula may make it easier to prove your particular result.

This document is not about finding the pattern, but is about strategies for constructing the examples from which a pattern may be found[1]. We have tried to cover a lot of important strategies, but there are many more, and the reader is encouraged to find other ways to look at things.

Even the example above can illustrate some tricks. One could calculate 1, then $1 + 2$, then $1 + 2 + 3$, then $1 + 2 + 3 + 4$, et cetera, but it's much easier to calculate one from the next. Once you know that

[1] A *very* good resource for finding patterns is "The On-Line Encyclopedia of Integer Sequences." It can be accessed at `oeis.org`.

$1 + 2 + 3 + 4 = 10$, you can obtain the next term simply by adding 5 to the 10 since you've already calculated most of $1 + 2 + 3 + 4 + 5$ already. (This will be called "General Strategy 2" which we will introduce in Section 2.)

The examples below are roughly in order of difficulty. We will not, in general, solve the problems, although a complete answer may be given without much (or any) motivation.

There are a number of exercises in the text and working on them will help you improve your ability to calculate and organize data. Some take a long time, and if you see what's going on, it may not be worthwhile to complete all the details. Solutions to every exercise can be found in Section 11.

2 General Strategies

Here is a short list of strategies that we will use in the examples in the following sections. It the idea behind a strategy isn't clear at first, don't worry: in the following sections when we use one of these strategies, we will remind you. In any case here is the list:

1. **Start with the smallest/easiest examples:** If the general problem has both small and large versions, work on the smallest ones first. For example, if your goal is to count the number of rearrangements of n things, don't start with $n = 10$. In fact, don't start with $n = 3$; start with $n = 1$ (or even $n = 0$!) Sometimes a problem is not general, but asks about a case where there are, say 100 objects. If you don't know how to proceed, start working on easier problems where $n = 1$ or 2 or 3.
2. **Look for patterns and reuse data you have already calculated:** Sometimes you can see how to build up a more complex example from the ones you have already worked out in simpler cases. We did this in the example in the introduction. In that case it was pretty obvious, but sometimes it is not so obvious.
3. **Divide the problem into cases:** If you're trying to count a complex set of patterns, see if you can break the set down into non-overlapping cases where it is easier to count the items in each case and then combine the results.
4. **Try to organize your results to avoid duplications or omissions:** Maybe you're trying to make a complete list of possibilites. You have a long list but you're not sure if you left something out, or if you made a mistake and your list includes a duplicate.
5. **When counting patterns, use "names" that have an intrinsic order:** In other words, if you have "4 items" let the "items" be A, B, C, and D, or perhaps 1, 2, 3 and 4. This will help in you do the organization mentioned in the strategy above.
6. **Look for patterns in the numbers you get:** Often you see lists of numbers that are familiar, like $1, 2, 4, 8, 16$ (the powers of 2) or $1, 3, 6, 10, 15$ (the triangular numbers) or $1, 1, 2, 3, 5, 8$ (the Fibonacci numbers) or $1, 1, 2, 6, 24, 120$ (the factorials). If you see a pattern and your next count doesn't match it, you should probably check your work.

7. **Cross check your answer:** Sometimes there are ways to sanity-check your answer. For example, if you are counting different cases for a problem to discover how many of each there are, but you know the total number, make sure your counts of the smaller cases add to the known total.

In the text that follows we will refer to these as "General Strategies," so if we're referring to strategy 3 above we'll refer to "General Strategy 3."

3 First Example: Counting Rearrangements (Permutations)

The general problem is this: given n items, in how many ways can they be rearranged? There are various interpretations of this problem: Are all the items different? If not, how many duplicates are there? To be a "rearrangement" does it mean that every item must move?

The term that mathematicians usually use for "rearrangement" is "permutation."

We'll consider different interpretations in the following subsections.

3.1 Rearrangements Without Duplicates

In this section we will consider rearrangements of n items, all of which are different. The term "rearrangement," however, will mean any arrangement of the items, even the one where they are left in their original order.

What we want to do here is to make lists of all possible rearrangements (permutations) of n different objects for small values of n.

We will begin by trying to count the permutations of four items by listing all of them and then counting the number we found. Although, as we will see, $n = 4$ seems small, it's almost always a better idea to begin with the smallest number, in this case, 1, or even 0 items (General Strategy 1).

If you try to count rearrangements even for $n = 4$ items and do it in a disorganized way, it is easy to get into trouble. For example, suppose the four items are A, B, C, and D. (Here we invoke General Strategy 5.) When you start a list like this:

$$\begin{array}{cc}
ABCD & DBAC \\
BCDA & CABD \\
CDBA & DABC \\
BACD & BDAC \\
ADCB & ACBD \\
CABD & ABDC \\
CDAB & DCAB
\end{array}$$

you are just asking for trouble. Although the list above contains only 14 rearrangements, it is difficult to find new ones. Each new one needs to be checked against all the ones generated so far to make sure there is no duplicate, and when do you stop? Just because you don't seem to be able to find another one doesn't guarantee that there isn't one. And in fact, the table above already contains an error! There are two copies of $CABD$ so the list really only has 13 items.

Even if the goal were only to make a complete list for $n = 4$ it still might be easier to start with smaller values of n, but let's first just do it the hard way. How can we make a list where we're sure we've omitted nothing? Here is our first use of General Strategy 4: Organize your results to avoid duplicates and omissions.

One nice way to start is simply to decide to list the candidates in alphabetical order. (One advantage of choosing the first four letters in the alphabet is that they (and sequences of them) do have a natural (alphabetical) ordering.) It's easy to see what comes first: $ABCD$.

What comes econd? Well, we'd like it to be as close to $ABCD$ as possible, and we can't start with ABC since the fourth one would then be D and we get a duplicate. But we can leave the AB part intact, and in order to avoid duplication, the only possibility is $ABDC$. What's next? Well, after the AB we have both possibilities: CD and DC, so we have to change something in the leading AB and a little consideration shows that the best we can do is begin with AC. To be as small as possible (in alphabetical order) we have to have $ACBD$ and then $ACDB$. Next go to $ADxx$, and so on.

Exercise 3.1. *Here are the first five permutations of the letters $ABCD$ in alphabetical order. Continue the list to the end. Hint: you should find 24 of them. See Section 11 for the solution.*

$$\begin{array}{c} ABCD \\ ABDC \\ ACBD \\ ACDB \\ ADBC \end{array}$$

If you're not good with the alphabetical ordering of letters, maybe it would have been easier to have the four items be the digits 1, 2, 3, and 4 (General Strategy 5). Then what we want to do is list them in numerical order which may be easier to see.

Exercise 3.2. *Here is the same exercise as above, but using digits instead of letters as the items to be rearranged. You've got the first 5; find the next 19 patterns.*

$$\begin{array}{c} 1234 \\ 1243 \\ 1324 \\ 1342 \\ 1423 \end{array}$$

Just looking at the complete solutions to the exercises above (the solutions appear in Section 11) reveals another way to think about it: First, let's list all the patterns that begin with A (or 1). Then with B, C and D in a similar way. But when we're working on the patterns that begin with A, let's do the ones whose second letter is B first, then those with C, and finally those whose second letter is D. This way you get a sort of automatic alphabetical ordering of the rearrangements. The more you practice this, the faster you get.

You also note that there are the same number of rearrangements that begin with each of the four letters (or numbers). This is sort of obvious, since once you've picked the first letter, you need to make all possible rearrangements of three different letters and there are the same number of these (6 in this case) that only depend on the fact that there are three *different* letters remaining to rearrange.

But perhaps a better way to begin (especially if the problem is not for a particular number of letters, but is to find out how many rearrangements there are with n letters) is to start with the simplest possible situations and list them. This is General Strategy 1

With just one: A, we have one rearrangement: A.

With A and B we've got two: AB and BA.

With three, we can do the alphabetical ordering trick and obtain these 6:

$$\begin{array}{c} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{array}$$

If you like, you can even consider the number of rearrangements of zero objects. This *is* one way to do it: a list with zero items. It is often a *very* good idea to consider the simplest (empty) case. It is often a "free" data point.

So with three (or four, if we consider the rearrangements of an empty set) we have the numbers $1, 2, 6$ (or $1, 1, 2, 6$). Using General Strategy 6 we may notice that these are the factorials: $0! = 1, 1! = 1, 2! = 2, 3! = 6$, so maybe we should be looking for $4! = 24$ examples when we count patterns with four items.

In fact, that the number of rearrangements of n items is given by $n!$, where:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1.$$

In other words, $n!$ (pronounced "n factorial") is obtained by starting at n and multiplying it by all the whole numbers less than n but greater than zero. There is a special case that makes life easy for mathematicians, and that is to define $0!$ ("zero factorial") to be 1.

A mathematician might think of it as follows: If I already know that there are $4!$ ways to rearrange 4 items, how many ways can I rearrange 5? Well, I can pick the first item in 5 different ways and for each of those 5, I know there are $4!$ ways to finish off the list. So the result will be

$$5 \cdot 4! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$$

ways. But there's really nothing special about going from 4 to 5. If I know that $n!$ is the number of rearrangements of n items then to count the rearrangements of $n + 1$ items, with the same reasoning, I'll obtain:

$$(n + 1) \cdot n! = (n + 1) \cdot n \cdot (n - 1) \cdots 3 \cdot 2 \cdot 1 = (n + 1)!$$

3.2 Rearrangements With Duplicates

In Section 3.1 we learned that if there is a list of n distinct items, there are $n!$ ways to rearrange them. What if some of the items are identical?

Let's try some examples. With three items, but with two the same, say AAB, there are 3 rearrangements instead of 6: AAB, ABA, and BAA.

With four items and two duplicates, there are 12 instead of 24. Check that the following list (in alphabetical order) covers all 12:

$$AABC,\ AACB,\ ABAC,\ ABCA,\ ACAB,\ ACBA,$$
$$BAAC,\ BACA,\ BCAA,\ CAAB,\ CABA,\ CBAA.$$

Here are a couple of exercises. Solve them and then check the answers in Section 11.

Exercise 3.3. *Work out the number of rearrangements of* 4 *items with* 3 *the same (*$AAAB$*).*

Exercise 3.4. *How about* 4 *items with* 2 *pairs of* 2 *(*$AABB$*).*

Exercise 3.5. *Try to find the number of rearrangements of* 5 *items there are if* 3 *are the same, say,* $AAABC$*.*

If you worked the three exercises above, you may notice that in all the cases we've checked, the number of rearrangements can be calculated by dividing the $n!$ that would be correct if all the items were different by appropriate numbers:

- For AAB there are 3 instead of the 6 had the letters been different.
- For $AABC$ there were 12 instead of 24.
- For $AAAB$ there were 4 instead of 24.
- For $AABB$ there were 6 instead of 24.
- For $AAABC$ there were 20 instead of 120.

For every pair of duplicates, divide by $2! = 1$; for every set of three duplicates, divide by $3! = 6$, and in general that pattern continues. Here would be the answer, for example, for the number of rearrangements of the word $MISSISSIPPI$. There are 11 letters in total, but 4 S's, 4 I's and 2 P's. The grand total turns out to be:

$$\frac{11!}{4! \cdot 4! \cdot 2!} = 34650.$$

3.3 Derangements

In the rearrangements we have considered so far, the final positions of the items could be anywhere. If we considered the items to be in order at the start, say as $ABCD$ or as 1234 then a rearrangment to $BDCA$ was one of the possibilities. Notice in this last example that C remained where it was, in third position. In fact we even considered it to be a "rearrangement" if everything stayed in its original position.

What we want to count now are special rearrangements where none of the items finishes in its original position. Such a rearrangment where everything moves is called a "derangement."

Here is a list of all the derangements for lists of size $1, 2$, and 3:

$$\begin{aligned} 1 &: \quad \text{None} \\ 12 &: \quad 21 \\ 123 &: \quad 231,\ 312 \end{aligned}$$

Exercise 3.6. *Try to find the number of derangements of* 4 *items. Hint: there are* 9 *of them.*

With longer and longer lists of items, it seems likely that at least one of them will wind up where it started in a random shuffling of the items, and in fact, this is true. As the number of items gets larger, the probability that a random rearrangement is in fact a derangement rapidly approaches a fixed probability, which is $1/e$, where e is the base of natural logarithms: $e \approx 2.71828182845\ldots$.

The probability of a non-derangement is thus $1 - 1/e \approx .63212055$. So if $50,000$ people randomly fill a stadium with $50,000$ seats without looking at their tickets, there is about a 63% chance that at least one person will be in his/her correct seat.

Here is an article on derangements:

`https://en.wikipedia.org/wiki/Derangement`

Exercise 3.7. *Try to find the number of derangements of* 5 *items.*

4 How Many Binary Numbers Are There of Length n?

Binary numbers are just sequences of 0's and 1's. If we wish to find the answer for all n it's best to begin using General Strategy 1. For $n = 1$ and $n = 2$ it's pretty easy. For $n = 1$ there are 2:

0
1

For $n = 2$, it's easy to find all 4:

00
01
10
11

For $n = 3$, already it's time to start being careful and to use General Strategy 4. List them in "numerical" order:

000	100
001	101
010	110
011	111

If we compare this table with the one above, it's easy to see a pattern (General Strategy 2). Note that both columns are the same (and identical to the previous table) if we leave out the first digit. In the first column, we put a 0 in front of all possible ways to complete the pattern and in the second, we add a leading 1. To work out the answer for $n = 4$, we just need to put a 0 in front of all 8 possibilities above, and then do the same thing with a leading 1:

0000	0100	1000	1100
0001	0101	1001	1101
0010	0110	1010	1110
0011	0111	1011	1111

Clearly the number of examples double with each additional digit, so for $n = 1, 2, 3, 4, \ldots$ we have $2, 4, 8, 16, \ldots$ possibilities.

The formula for general n is simply 2^n.

5 Choosing k Things from a Set of n Different Things

For this set of problems, when we pick the k things, the order does not matter. In other words, if we've picked ABC as three items, it is no different from ACB, CAB or any of the other three rearrangements. Only one of the six rearrangements should be in the final list. A practical example of this problem might be, if you've got $n = 10$ people on your basketball team, how many different combinations of $k = 5$ of them could you put on the court?

In mathematics, these numbers are called "combinations" or "binomial coefficients".

Beginning with a concrete example that's too big (since we should probably begin with General Strategy 1), we will count the number of ways to choose 3 items from a set of 5.

Using General Strategies 5 and 4 we can use A, B, C, D, and E as our items and list the sets in alphabetical order. (Also, within a set, we list the letters in alphabetical order: if the set includes D, E and B, we'll write it down as BDE in order to avoid duplicates in our final list.) We begin by listing all of the combinations that have A as the earliest letter in the alphabet, then those with B as the earliest, and so on. As we're listing each subgroup, use the same strategy on the subset. Here's a list of all 10 possibilities using that strategy (reading down the first column, then moving to the second):

$$\begin{array}{cc} ABC & ADE \\ ABD & BCD \\ ABE & BCE \\ ACD & BDE \\ ACE & CDE \end{array}$$

Notice that when we're done with the sets that contain A, we will never see another A again, so when we begin listing the B combinations, we only need to consider groups beginning with the letters C and beyond.

Exercise 5.1. *Make a list, as above, to find all of the ways to choose* 4 *items from a set of* 6*. In this case, let the items be the digits* 123456*. So as not to omit anything, list your answers in numerical order.*

In Section 3 we wanted to find a single result for every number n; namely, the number of ways to rearrange n items. Here the problem is more complex: we will have, for each n, results for $k = 0, 1, 2, \ldots, n$. In other words, for each n, we have $n + 1$ different answers.

A good way to organize this is as a 2-dimensional table where the rows correspond to different values of n and the columns, to different values of k. It will not be a complete table, since in row 3, for example, we will have entries only for values of k equal to 3 or less[2].

General Strategy 1 tells us to look at small, easy-to-calculate examples. There are actually an infinite number of very easy examples. If we have n items, there is only one way to take all n of them, and this is true for every n. Similarly, and it's worthwhile to convince yourself that this is true, there is also (for every n) exactly 1 way to pick zero items: if you are asked to do so, the one thing you can do is to present an empty list.

So with almost no work at all, we have the beginnings of our table (where we have put the letter "x" in all the slots where we will want numbers but have not counted them yet. Note that we did work out above the case where $n = 5$ and $k = 3$.):

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	x	1					
3	1	x	x	1				
4	1	x	x	x	1			
5	1	x	x	10	x	1		
6	1	x	x	x	x	x	1	
7	1	x	x	x	x	x	x	1

Actually, there is another infinite set of results that is easy to obtain. If we have n items, it is clear that there are exactly n ways to choose 1 of them. That's pretty obvious, but this means that there are also exactly n ways to choose $n - 1$ of them. That's because if we choose $n - 1$ of them, we are effectively choosing the 1 item to be left out, and there are obviously n ways to do that. Including that observation, our table of results looks like this:

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	x	4	1			
5	1	5	x	10	5	1		
6	1	6	x	x	x	6	1	
7	1	7	x	x	x	x	7	1

In fact, that last observation (that there are the same number of ways to pick 1 item is the same as the number of ways to leave out 1 item) can be generalized if we note that the number of ways to choose 2 items is the same as the number of ways to omit 2 items and so on. In general, the number of ways to pick k items must be the number of ways to omit k items, so, for example, our work to count the number of ways to pick 3 items from a set of 5 is the same as the number of ways to omit 3 items (which is the number of ways to choose 2 items. Taking into account this observation, our table looks like this:

[2] We could actually just put zeros in these columns and that would make sense, too, since there are zero ways to pick 4 items from a set of 3, but in what follows we'll just leave columns of that sort blank.

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	x	4	1			
5	1	5	10	10	5	1		
6	1	6	y	z	y	6	1	
7	1	7	v	w	w	v	7	1

where we have replaced some of the letters "x" by other ones. In other words, since there are two w's, for example, each slot in the table where a "w" appears must contain the same number.

Let's work out the cases $n = 4$ and $k = 2$ as above, yielding 6 examples:

$$\begin{array}{cc} AB & BC \\ AC & BD \\ AD & CD \end{array}$$

And now for $n = 6$ and $k = 2$, yielding 15 examples:

$$\begin{array}{ccc} AB & BC & CE \\ AC & BD & CF \\ AD & BE & DE \\ AE & BF & DF \\ AF & CD & EF \end{array}$$

Filling these values into our table we have:

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	z	15	6	1	
7	1	7	v	w	w	v	7	1

If you don't recognize Pascal's triangle above, skew the table above to the right so that the top 1 is centered and here's what you get. Note that other than the 1's running down the outsides of the triangle, every other number is just the sum of the two numbers immediately above it to the right and left. Here is the skewed version:

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&2&&1&&&&& \\
&&&&1&&3&&3&&1&&&& \\
&&&1&&4&&6&&4&&1&&& \\
&&1&&5&&10&&10&&5&&1&& \\
&1&&6&&15&&\mathrm{z}&&15&&6&&1& \\
1&&7&&\mathrm{v}&&\mathrm{w}&&\mathrm{w}&&\mathrm{v}&&7&&1
\end{array}$$

In fact, that's the answer to the general problem. Here's what the completely-filled table (up to $n = k = 7$) looks like:

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Here is a nice way to see that the method for filling Pascal's triangle works (where the add the two numbers above to make the number below). We will use specific numbers, but it should be obvious that this argument will work for any n and k. Suppose we have worked out everything down to row 6 and we want to find the number of ways to choose 5 things from a set of 7. Suppose the 7 items are A, B, C, D, E, F, and G. In the 5 items we pick, if G is in that set, we only need to pick 4 more, so the set can be completed by choosing 4 from the remaining 6 and the row above tells us there are 15 ways to do it. If the G is not included in our choice, we need all 5 items to be chosen from among A and F, and there are (according to the table in row 6) exactly 6 ways to choose 5 things from a set of 6. To count the total, we just add the ways to make our set using the G (15 ways) and the ways to make it not using the G (6 ways) for a total of $15 + 6 = 21$ ways, and we see that there's a 21 in that slot.

The formula for the number of ways to choose k things from a set of n which is usually pronounced as "n choose k" is as follows:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

remembering, of course that $0! = 1$. It's worthwhile to check some of the table entries to verify that this formula is correct. (Remember to number the rows and columns starting at zero.)

Exercise 5.2. *Complete Pascal's triangle to row* 10 *(well, actually to the* 11^{th} *row, since we usually call the first one row zero).*

6 In How Many Ways Can You Partition the Number n?

By partitioning a number we mean writing it as a sum of non-increasing whole numbers. For example, we can partition the number 5 in the following 7 ways:

$$\begin{array}{l}1+1+1+1+1\\2+1+1+1\\2+2+1\\3+1+1\\3+2\\4+1\\5\end{array}$$

We would like to find an organized way to list the partitions of the first few numbers. The table above may provide some ideas for how to do it. Here we've listed first the partition that consists of only 1's, then partitions that use 2's or smaller, then 3's or smaller, et cetera.

Exercise 6.1. *Try to find all the partitions for the numbers* 0 *through* 9*. Here are the counts you should find (starting from zero, where the only partition of zero is the empty formula):*

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30.$$

Keep track of the strategies that you used.

One idea is that when you have a complete set for a particular number, you can just add a "$+1$" to each row to get many of the partitions for the next larger number. Then you have to search for additional partitions that can be constructed by collapsing the extra 1's in various ways.

For example, after adding a $+1$ to each of the partitions of 5, we see that $2+2+1+1$ can also look like $2+2+2$, $3+1+1+1$ looks a bit like $3+3$, $4+1+1$ can convert to $4+2$ and finally note that we need to add a 6 at the end for four additional partitions. This gives 11 total partitions for 6, but as you can see, you've got to be careful!

Here's an improvement over that idea. Make a 2-dimensional table of values with n running vertically from 1 and k running horizontally, also from 1. The number in the n^{th} row, k^{th} column is the number of ways to partition n using values that are k or smaller. Check some small examples to verify that this is the case.

The largest number in any row n is the total number of partitions of the number n, and will appear for the first time in the n^{th} column. That's because there will be no partitions using numbers $n+1$ and larger. In any case, here is a table for the first few integers:

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3	3
4	1	3	4	5	5	5	5	5	5	5
5	1	3	5	6	7	7	7	7	7	7
6	1	4	7	9	10	11	11	11	11	11
7	1	4	8	11	13	14	15	15	15	15
8	1									

Let's check this table against the 11 partitions of 6:

$$\begin{array}{l}
1+1+1+1+1+1 \\
2+1+1+1+1 \\
2+2+1+1 \\
2+2+2 \\
3+1+1+1 \\
3+2+1 \\
3+3 \\
4+1+1 \\
4+2 \\
5+1 \\
6
\end{array}$$

There is 1 partition of 6 using only 1 (and this is true of every number). There are 3 that begin with 2, so $4 = 3 + 1$ that begin with 2 or less. There are 3 more that begin with 3, so there are $7 = 3 + 3 + 1$ total that begin with 3 or less. Two more begin with 4 for a total of $9 = 2 + 3 + 3 + 1$, one more with 5 and one final one that begins with 6.

The numbers in row n and column k are the total number of partitions of n. We might as well continue that last number forever, since there are no additional ways to partition n using numbers that begin with $n + 1$ or larger.

Now let's see how to fill in row number 8, using the already-calculated data in the table which is complete up to 7 (General Strategy 2). We will work our way across the 8^{th} row one step at a time:

1. In the first row there is always a 1, since there is only one way to partition a number using only 1. In this case, it'll be $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$.

2. If we begin the partition of 8 using a 2, we need 6 more, and we have to complete it using only partitions of 6 that use 2 or less. That's the number in row 6, column 2, which is 4. We need to add those 4 ways to the 1 way that we can complete with 1 (or less), so a 5 goes in that slot.

3. In addition to the 5 from the previous column, if we begin the partition with a 3, we need 5 more and the table tells us that there are 5 ways to partition 5 using 3 or less. Thus we get a 10 in this slot.

4. If we begin with a 4, there are 5 ways to partition 4 with 4 or fewer, so the total for this position is 15.

5. Beginning with 5, there are 3 left, and 3 ways to partition 3, for a total of 18.
6. Beginning with 6, there are 2 ways to partition the remaining 2, for a total of 20.
7. Beginning with 7, there's only 1 way left, so we've got 21.
8. Finally, beginning with 8, we're already done, so the final total is 22. The rest of the columns will contain 22 since there are no additional ways to make an 8.

Here is the table, complete to one more row:

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3	3
4	1	3	4	5	5	5	5	5	5	5
5	1	3	5	6	7	7	7	7	7	7
6	1	4	7	9	10	11	11	11	11	11
7	1	4	8	11	13	14	15	15	15	15
8	1	5	10	15	18	20	21	22	22	22

There aren't any nice closed-form formulas to count partitions, but there are some recursive formulas that do the job. Here is one. Suppose you multiply the infinite series:

$$(1-x)(1-x^2)(1-x^3)(1-x^4)...$$

We obtain:

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots$$

If $p(n)$ is the number of partitions of n and if $p(m) = 0$ when $m < 0$ we have:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - \cdots$$

This is not a particularly nice formula, but it does work.

Here is the Wikipedia article on partitions:

`https://en.wikipedia.org/wiki/Partition_(number_theory)`

7 Calculate the Sums of Rolled Dice

If you roll n dice and add the numbers on the faces, how many ways are there to obtain a total of k? In this section, we will always consider only standard 6-sided dice with the numbers 1 through 6 on the faces.

This is another problem that has two variables: the number n of dice and the total, k that the numbers on the dice should add to. We probably want to make a table with n and k and start filling in from the easy cases.

The easiest case is when $n = 1$, and then there is exactly one way to make each of the sums, 1 through 6.

With two dice, the possible sums are now 2 though 12, with 3 dice, 3 through 18 and so on. With k dice, the possible sums run from k (all 1's) through $6k$ (all 6's).

First, imagine that each die is a different color so we can see exactly what is happening. So for example, with two dice, there is only one way to obtain a 2; namely, both the red and green die have to show 1. To obtain a 3, however, there are two possibilities: 1 on red and 2 on green or 2 on red and 1 on green.

Perhaps the first thing to note is that with one die, there are $6^1 = 6$ outcomes. With two dice, there are $6^2 = 36$ outcomes. With three dice, $6^3 = 216$ outcomes, and with k dice, there are 6^k possible outcomes.

Let's look at the two dice case first. Here is a table showing the outcomes (sums) for all 36 possible pairs of outcomes:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

If we look at the table above, it's clear that there is 1 way to obtain a 2, 2 ways to obtain 3, 3 ways to get 4, and so on, up to 6 ways to get 7. Then the possibilities decrease to just 1 way to obtain 12. Here is a table:

Total	2	3	4	5	6	7	8	9	10	11	12
Counts	1	2	3	4	5	6	5	4	3	2	1

One difficulty that arises is that for 3 dice the "table" really should be a 3-dimensional one, and for 4 dice, a 4-dimensional one which can be a bit difficult to draw.

For even 3 dice, even listing out all the possibilities (although straightforward) would be burdensome. There are $3^3 = 216$ possibilities for the outcomes of three dice. For 4 dice, there are $6^4 = 1296$ possibilities[3].

Exercise 7.1. *Make a list of all the ways to have a total of* 10 *from three dice. Hint: here are* 27 *ways to do it. Remember that* 631 *is different from* 316*, et cetera, because we are imagining having dice of different colors so the numbers can appear on the dice in any order and each represents an outcome of a possible throw.*

Let's look at the situation with 3 dice and see what we can work out. General Strategy 1 tells us to begin with the easiest possibility, which is a total of 3, and 3 can be obtained in only one way: a 1 on each of the three dice.

A little thought shows that there are 3 ways to obtain a total of four: 112, 121, and 211. To obtain a total of 5, we have: 311, 131, 113, 221, 212, and 122, for a total of 6 ways. This approach will work, but

[3] We will not do it here, but perhaps another good approach to this problem might be to think about simpler "dice": dice that have only 1 face, or 2 faces, et cetera.

notice that to continue, we need to find all 216 arrangements, which will give us plenty of opportunities for error. It would be nice to have a better way to count.

One useful observation is this. There is a symmetry in the outcomes in the sense that there is 1 way to obtain the extreme results: a total of 3 has to be 111 and a total of 18 has to be 666. For totals of 2 or 17, the three dice (in some order) have to be either 112 or 665. Each step up from the bottom or down from the top lets in exactly the same number of possibilities. It seems obvious (although we will show it below) that the number of possibilites builds to a maximum and then decreases to 1 way as we count possible ways to obtain totals of 3 through 18. There are 16 possible sums, so the resulting numbers will ramp up for 8 and then will ramp down in exactly the same way. Using variables to indicate the number of possibilities for the so far unknown counts we will have something like this where the same letter represents the same (so far unknown) number:

Total	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Counts	1	3	6	a	b	c	d	e	e	d	c	b	a	6	3	1

When we're done, we will have a nice way to check as well, since all the counts need to add to 216, we know that $2(1+3+6+a+b+c+d+e) = 216$ or $1+3+6+a+b+c+d+e = 108$.

Look at the count for a total of 6 (and for 15, by symmetry). Here are the ways to obtain a total of 6 (ignoring which dice have the numbers): 411, 321, 222. There are three ways to obtain the first outcome (the single 4 can be anywhere) there are 6 ways to obtain 321 (this is just a count of the rearrangements of 3 items, which we found to be 6 in Section 3), and only one way to obtain 222 (all three dice the same; namely, 2. So $a = 3+6+1 = 10$.

To obtain 7 here are possible dice outcomes: 511, 421, 331, and 322. Using the arguments in the previous paragraph, there are 3, 6, 3, and 3 of these respectively, for a total of $b = 15$.

For 8 we have: 611, 521, 431, 422, and 332, yielding $c = 3+6+6+3+3 = 21$.

For 9 we have: 621, 531, 522, 441, 432, 333, yielding $d = 6+6+3+3+6+1 = 25$.

Finally, for 10 we have: 631, 622, 541, 532, 442, 433, yielding $e = 6+3+6+6+3+3 = 27$.

Check the results: $1+3+6+a+b+c+d+e = 10+10+15+21+25+27 = 108$. Here's the final table for 3 dice:

Total	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Counts	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1

It turns out that there is no "nice" formula for the number of ways to obtain the number of ways a total of k can be obtained by throwing n dice. A discussion is available here:

`http://mathworld.wolfram.com/Dice.html`

With 4 dice the situation becomes quite complex. Totals can range from 4 to 24, but as we saw above, the number of ways to obtain 4 is the same as the number of ways to obtain 24, the number of ways to obtain 5 is the number of ways to obtain 23, et cetera.

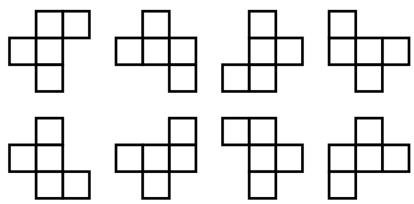
Exercise 7.2. *Verify some of the numbers of ways to obtain totals between* 4 *and* 14 *using four dice. The solution in Section 11 contains not only the totals, but the individual numbers of ways, for example, to obtain a total of* 10 *with the pattern* 5221*.*

8 Count the n-Square Polyominoes

A Polyomino is a figure made by attaching squares together along their sides so that the resulting figure is completely connected and lies flat on a plane. The sides are connected from corner to corner and to be connected, two squares must be connected along an entire side; not just at a corner.

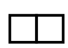
The problem we'd like to consider here is to count the number of different polyominoes there are that consist of exactly n squares. We will consider two of them to be the same if they can be made to match by rotating them, translating them, or flipping them over (or if you prefer, taking a mirror image).

For example, the following 8 pentominoes are all to be considered to be the same, since every one is a rotation and/or reflection of every other one.

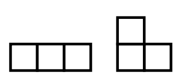


There is only one monomino (polyomino made from a single square — it's not really a *poly*omino, is it?):

Everyone is probably familiar with the single domino made from two squares:



Here are the two different trominoes (made with three squares):



Now try your hand at finding polyominoes. Keep track of the strategies you use.

Exercise 8.1. *Find all the tetrominoes (4-square polyominoes).*

Exercise 8.2. *Find all the pentominoes (5-square polyominoes).*

Polyominoes (especially the pentominoes) are used in many puzzles and games. The name was invented by Solomon W. Golomb in 1953.

Here are a couple of strategies you might use to make sure you have a complete list of the polyominoes of a given size. Perhaps you have even better ones:

1. To avoid duplicates think of ways to divide them into groups. For example always arrange them so that the longest dimension is horizontal and the shorter (if one is shorter) is vertical. Then arrange the polyomino so that as many as possible of the squares are on the low side of the bounding rectangle, and also as many as possible on the left.
2. If you are looking for, say, hexominoes (6-square polyominoes) look for them in rectangles of sizes 1×6, 2×5, 2×4, 3×4, 3×3, and 2×3.
3. If you are sure you have all the solutions for a given size, and you want to work on the next size up, take each of those and add a square at every possible point. This will give a huge number of duplicates, but a lot of the additions can be done mentally.

Exercise 8.3. *Find all the hexominoes (*6*-square polyominoes). Hint: there are* 35 *of them.*

Here is some information about the polyominoes:

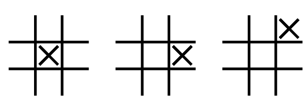
`https://en.wikipedia.org/wiki/Polyomino`

9 Tic-Tac-Toe Positions

Special thanks to Marc Roth for suggesting this set of problems!

Since every rotation or mirroring of a standard 3×3 tic-tac-toe position is basically equivalent, when X makes the first move, there are effectively only 3 different moves X can make: in the center, in a corner, or on an edge. (These are clearly different moves since there are 4 possible wins with a line through the center, 3 possible wins with a line through a corner, and only 2 possible wins with a line through an edge.

Here are all possible first moves by X:



Exercise 9.1. *Find all possible positions after two moves (*X *followed by* O*).*

Exercise 9.2. *Imagine you are playing "*X *only tic-tac-toe.[4]" (In other words, only* X *gets played.) How many positions are there after* 2 *moves? How did you organize your answers?*

In the solutions to the following problems is a suggestion about how to make sure you have all possible positions and no extras. There are other strategies; maybe you can think of additional ones, but here is one that is not quite as good:

[4] There is such a game: `https://miseregames.files.wordpress.com/2012/04/x-onlyttt.pdf`.

Classify them by the number of X's on each of the three rows. You will have to be careful about duplicates. Also note that due to rotation and mirroring, if you've found all the solutions with one on the top row and two in the middle, for example, there is no need to search for solutions with two in the middle row and one on the bottom.

Exercise 9.3. *Imagine you are playing "X only tic-tac-toe." (In other words, only X gets played.) How many positions are there after* 3 *moves?*

Exercise 9.4. *Imagine you are playing "X only tic-tac-toe." (In other words, only X gets played.) How many positions are there after* 4 *moves?*

Exercise 9.5. *Imagine you are playing "X only tic-tac-toe." (In other words, only X gets played.) How many positions are there after* $5, 6, 7$*, and* 8 *moves?*

10 Additional Problems

In this section we will just present exercises similar to what has been presented above, but without the discussion. Solutions will appear, as before, in Section 11.

Exercise 10.1. *Balanced parentheses: If you have one pair of parentheses, there is only one way to arrange them so that the result is balanced:* $()$*. With two pairs, there are two ways to do it:* $(())$ *and* $()()$*. With three pairs, we have:* $((()))$*,* $(()())$*,* $(())()$*,* $()(())$ *and* $()()()$ *for a total of* 5 *ways. Find all balanced sets of parentheses eith* 4 *pairs and with* 5 *pairs.*

Exercise 10.2. *Suppose you want to make a beaded necklace with black and white beads. Basically, you will have a ring of beads, and the idea is to find out how many different patterns there are with B black beads and W white beads. Note that all the rotations of a necklace are the same, and that a necklace can be flipped over so that the clockwise/counterclockwise orientation of the beads is flipped.*

If we imagine cutting the necklace at some point, we can then arrange the beads from left to right, so (because of rotation) the following 5*-bead necklaces are really the same:*

$$BWWWW, WBWWW, WWBWW, WWWBW, WWWWB.$$

Similarly, the following two necklaces are the same, since one can be flipped over and rotated into the other:

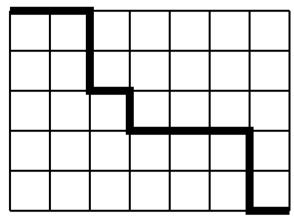
$$WBWWBBBB, WWBWBBBB.$$

In any case, create some tables that list the number of possible necklaces for small numbers of W and B beads. (Remember to include easy cases, including zero and one bead of the colors.)

Exercise 10.3. *Suppose you have standard* 2*-square dominoes and you wish to cover a strip that is* 2 *squares wide and n squares long. If $n = 1$ there is obviously only one way to cover it. If $n = 2$, there are two ways. If $n = 3$ there are* 3 *ways. Can you make a table for larger values of n?*

Exercise 10.4. *Suppose your city has only north-south and east-west streets. Choose a grid that has m north-south streets and n east-west streets. In how many ways can you travel from one corner to the opposite corner of the grid in the shortest possible path?*

Here is a typical shortest path on a grid with 8 *north-south streets and* 6 *east-west streets:*



11 Solutions to Exercises

Exercise 3.1: Here is the complete list (read down each column, then go to the top of the next column to continue the list):

$$\begin{array}{cccc} ABCD & BACD & CABD & DABC \\ ABDC & BADC & CADB & DACB \\ ACBD & BCAD & CBAD & DBAC \\ ACDB & BCDA & CBDA & DBCA \\ ADBC & BDAC & CDAB & DCAB \\ ADCB & BDCA & CDBA & DCBA \end{array}$$

Exercise 3.2: Here is the solution using digits:

$$\begin{array}{cccc} 1234 & 2134 & 3124 & 4123 \\ 1243 & 2143 & 3142 & 4132 \\ 1324 & 2314 & 3214 & 4213 \\ 1342 & 2341 & 3241 & 4231 \\ 1423 & 2413 & 3412 & 4312 \\ 1432 & 2431 & 3421 & 4321 \end{array}$$

Exercise 3.3: Rearrangements of $AAAB$, 4 total:

$$AAAB,\ AABA,\ ABAA,\ BAAA.$$

Exercise 3.4: Rearrangements of $AABB$, 6 total:

$$AABB,\ ABAB,\ ABBA,\ BAAB,\ BABA,\ BBAA.$$

Exercise 3.5: Rearrangements of $AAABC$, 20 total:

$$AAABC,\ AAACB,\ AABAC,\ AABCA,\ AACAB,$$
$$AACBA,\ ABAAC,\ ABACA,\ ABCAA,\ ACAAB,$$
$$ACABA,\ ACBAA,\ BAAAC,\ BAACA,\ BACAA,$$
$$BCAAA,\ CAAAB,\ CAABA,\ CABAA,\ CBAAA.$$

Exercise 3.6: There are 9 of them:

$$2143 \quad 2341 \quad 2413 \quad 3124 \quad 3421 \quad 3412 \quad 4123 \quad 4312 \quad 4321$$

Exercise 3.7: There are 44 of them, arranged in numerical order:

21453	21534	23154	23451	23514	24153	24513	24531	25134	25413	25431
31254	31452	31524	34152	34251	34512	34521	35124	35214	35412	35421
41253	41523	41532	43152	43251	43512	43521	45123	45132	45213	45231
51234	51423	51432	53124	53214	53412	53421	54123	54132	54213	54231

Notice that after we have worked out the first line, we basically know that there will be 3 more similar copies.

Exercise 5.1: Here are the 15 ways to choose 4 items from a set of 6:

$$\begin{array}{ccc} 1234 & 1256 & 2345 \\ 1235 & 1345 & 2346 \\ 1236 & 1346 & 2356 \\ 1245 & 1356 & 2456 \\ 1246 & 1456 & 3456 \end{array}$$

Exercise 5.1: Here are the first 11 rows of Pascal's triangle:

$$\begin{array}{ccccccccccccccccccccc}
&&&&&&&&&&1&&&&&&&&&& \\
&&&&&&&&&1&&1&&&&&&&&& \\
&&&&&&&&1&&2&&1&&&&&&&& \\
&&&&&&&1&&3&&3&&1&&&&&&& \\
&&&&&&1&&4&&6&&4&&1&&&&&& \\
&&&&&1&&5&&10&&10&&5&&1&&&&& \\
&&&&1&&6&&15&&20&&15&&6&&1&&&& \\
&&&1&&7&&21&&35&&35&&21&&7&&1&&& \\
&&1&&8&&28&&56&&70&&56&&28&&8&&1&& \\
&1&&9&&36&&84&&126&&126&&84&&36&&9&&1& \\
1&&10&&45&&120&&210&&252&&210&&120&&45&&10&&1
\end{array}$$

Exercise 6.1: Here are the partitions of the numbers from 0 through 9. To save space, we omit the $+$ signs, so $4+3+1+1$ will be listed as 4311: For 0 (1 total):

$$\mathrm{Theemptylist}$$

For 1 (1 total):

$$1$$

For 2 (2 total):

$$11 \quad 2$$

For 3 (3 total):

$$111 \quad 21 \quad 3$$

For 4 (5 total):

$$1111 \quad 211 \quad 22 \quad 31 \quad 4$$

For 5 (7 total):

$$11111 \quad 2111 \quad 221 \quad 311 \quad 32 \quad 41 \quad 5$$

For 6 (11 total):

$$111111 \quad 21111 \quad 2211 \quad 222 \quad 3111 \quad 321 \quad 33 \quad 411 \quad 42 \quad 51 \quad 6$$

For 7 (15 total):

$$1111111 \quad 211111 \quad 22111 \quad 2221 \quad 31111 \quad 3211 \quad 322$$

$$331 \quad 4111 \quad 421 \quad 43 \quad 511 \quad 52 \quad 61 \quad 7$$

For 8 (22 total):

$$11111111 \quad 2111111 \quad 221111 \quad 22211 \quad 2222 \quad 311111 \quad 32111$$

$$3221 \quad 3311 \quad 332 \quad 41111 \quad 4211 \quad 422 \quad 431 \quad 44 \quad 5111$$

$$521 \quad 53 \quad 611 \quad 62 \quad 71 \quad 8$$

For 9 (30 total):

$$111111111 \quad 21111111 \quad 2211111 \quad 222111 \quad 22221 \quad 3111111$$

$$321111 \quad 32211 \quad 3222 \quad 33111 \quad 3321 \quad 333 \quad 411111 \quad 42111$$

$$4221 \quad 4311 \quad 432 \quad 441 \quad 51111 \quad 5211 \quad 522 \quad 531 \quad 54$$

$$6111 \quad 621 \quad 63 \quad 711 \quad 72 \quad 81 \quad 9$$

Exercise 7.1: Here are the 27 ways to obtain a total of 10 when rolling 3 dice.

631	613	361	316	163	136
622	262	226			
541	514	451	415	154	145
532	523	352	325	253	235
442	424	244			
433	343	334			

Exercise 7.2: Here is a complete table counting the number of ways to obtain various totals using 4 dice. It consists solely of the results for totals between 4 and 14, after which the numbers go back down in a symmetric way.

Total	Dice Faces atop Rearrangements												Arrangements
4	1111												
	1												1
5	2111												
	4												4
6	3111	2211											
	4	6											10
7	4111	3211	2221										
	4	12	4										20
8	5111	4211	3311	3221	2222								
	4	12	6	12	1								35
9	6111	5211	4311	4221	3321	3222							
	4	12	12	12	12	4							56
10	6211	5311	5221	4411	4321	4222	3331	3322					
	12	12	12	6	24	4	4	6					80
11	6311	6221	5411	5321	5222	4421	4331	4322	3332				
	12	12	12	24	4	12	12	12	4				104
12	6411	6321	6222	5511	5421	5331	5322	4431	4422	4332	3333		
	12	24	4	6	24	12	12	12	6	12	1		125
13	6511	6421	6331	6322	5521	5431	5422	5332	4441	4432	4333		
	12	24	12	12	12	24	12	12	4	12	4		140
14	6611	6521	6431	6422	6332	5531	5522	5441	5432	5333	4442	4433	
	6	24	24	12	12	12	6	12	24	4	4	6	146

Here is how to interpret the table above. The numbers down the left column correspond to the desired total of the four dice. Along each row are up to 12 ways that the desired sum can be achieved. Underneath each of those ways is a total number of ways to rearrange them among the four dice. For example, to obtain a total of 8 with the dice faces showing 4211 there are 12 ways the dice could come up to yield this particular set of numbers:

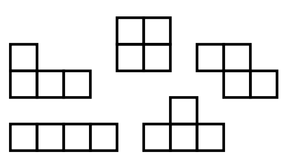
$$4211,\ 4121,\ 4112,\ 2411,\ 2141,\ 2114,\ 1421,\ 1412,\ 1241,\ 1214,\ 1124,\ 1142.$$

See Section 3.2 if you have difficulties calculating these numbers.

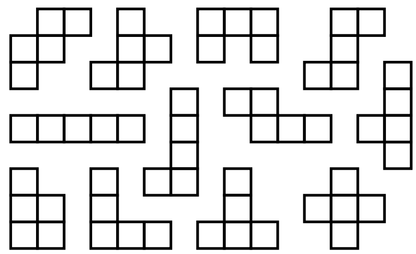
We do have an easy check see if at least our numbers are reasonable. We know that there are, in total, $6^4 = 1296$ outcomes, so all the numbers in the last column should add to that. Well, not quite, since we only went up to totals of 14, knowing that the number of arrangements should be symmetric. The value of 14 is in the center, so when we're adding, we should only include the 146 once, but twice for all the others. It is easy to check that:

$$2(1 + 4 + 10 + 20 + 35 + 56 + 80 + 104 + 125 + 140) + 146 = 1296.$$

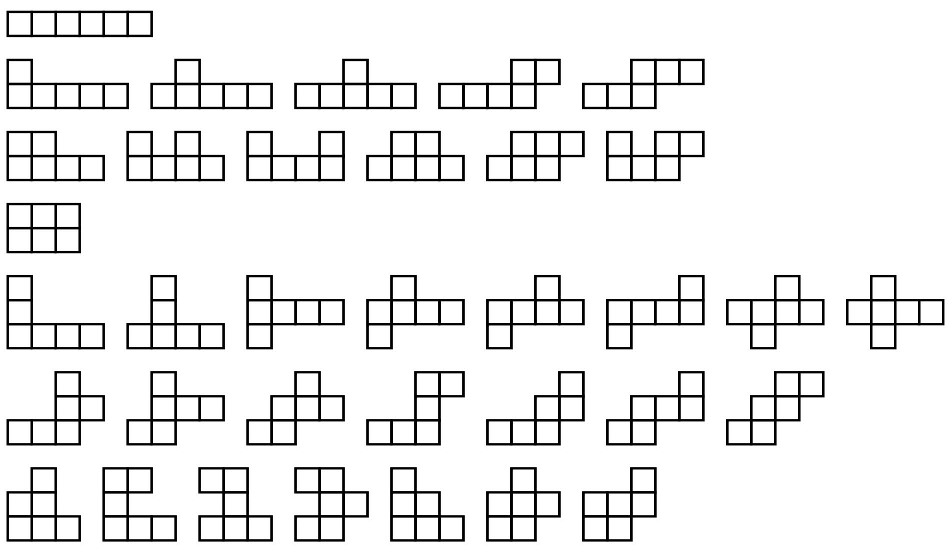
Exercise 8.1: Here are the 5 tetrominoes:



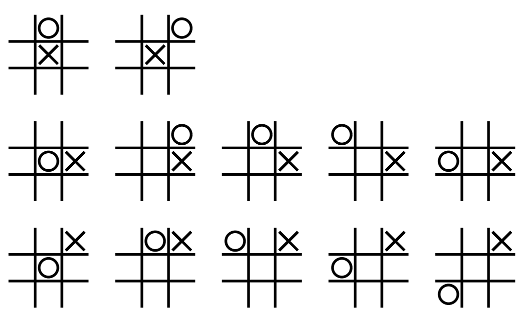
Exercise 8.2: Here are the 12 pentominoes:



Exercise 8.3: Here are the 35 hexominoes, arranged by bounding box size. The first row contains all (only one) of them that fits into a 1×6 bounding box. The second row, all of them that fit into a 2×5 box, then a 2×4 box, a 2×3 box. Next are two rows of hexominoes that fit in a 3×4 box and finally, a row of all those that fit in a 3×3 box.

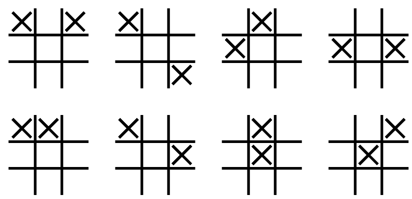


Exercise 9.1: Here is a list of all possible positions of a tic-tac-toe game after two moves:



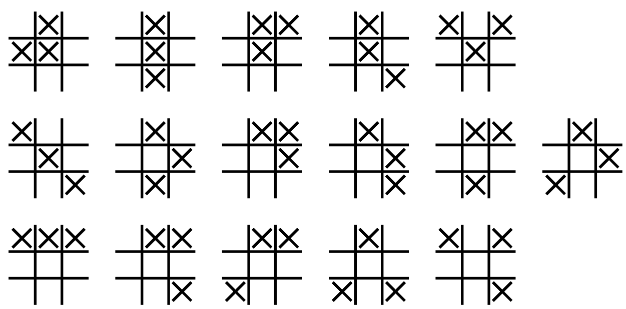
Perhaps the easiest way to organize this is to consider each of the possible O moves for each of the initial X moves. That's how the list above is arranged.

Exercise 9.2: Here is a list of all possible positions of a tic-tac-toe game after two moves:



The list above is arranged first with two corner moves, then with two edge moves, then with an edge and corner move, and finally, the two moves where one X is in the center.

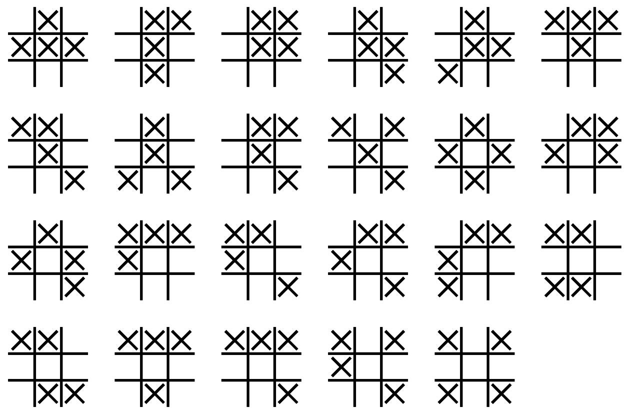
Exercise 9.3: Here is a list of all possible positions of a tic-tac-toe game after three moves:



One organization technique is to classify all the positions based on the number of X's in the center, corners and edges. Here's a complete list:

Center	Edge	Corner	Count
1	2	0	2
1	1	1	2
1	0	2	2
0	3	0	1
0	2	1	4
0	1	2	4
0	0	3	1

Exercise 9.4: Here is a list of all possible positions of a tic-tac-toe game after three moves:



Again, a reasonable organization technique is to classify all the positions based on the number of X's in the center, corners and edges. Here's a complete list. To avoid errors, it might be better to somehow split the case where there are two edges and two corners.

Center	Edge	Corner	Count
1	3	0	1
1	2	1	4
1	1	2	4
1	0	3	1
0	4	0	1
0	3	1	2
0	2	2	7
0	1	3	2
0	0	4	1

Exercise 9.5: This question may seem hard, but we've already done all the work in previous exercises. The number of ways to fill 5 slots is the same as the number of ways to fill 4 holes, which we solved in the previous example, yielding 23. For 6 spots it's the same as 3 holes; namely, 16. And so on.

Exercise 10.1: The solutions for general n (including $n = 0$) form the Catalan numbers. Here is a list of all possible valid arrangements of parentheses:

$n = 0$:	*	1 way
$n = 1$:	`()`	1 way
$n = 2$:	`()(), (())`	2 ways
$n = 3$:	`()()(), ()(()), (())(), (()()), ((()))`	5 ways

$n = 4$:	`()()()(), ()()(()), ()(())(), ()(()()), ()((())),`
	`(())()(), (())(()), (()())(), ((()))(), (()()()),`
	`(()(())), ((())()), ((()())), (((())))`
		14 ways

$n = 5$:	`()()()()(), ()()()(()), ()()(())(), ()()(()()), ()()((())),`
	`()(())()(), ()(())(()), ()(()())(), ()((()))(), ()(()()()),`
	`()(()(())), ()((())()), ()((()())), ()(((()))), (())()()(),`
	`(())()(()), (())(())(), (())(()()), (())((())), (()())()(),`
	`(()())(()), ((()))()(), ((()))(()), (()()())(), (()(()))(),`
	`((())())(), ((()()))(), (((())))(), (()()()()), (()()(())),`
	`(()(())()), (()(()())), (()((()))), ((())()()), ((())(())),`
	`((()())()), (((()))()), ((()()())), ((()(()))), (((())())),`
	`(((()()))), ((((()))))`
		42 ways

Exercise 10.2: Here is a table of the number of different necklaces that can be made with B black beads and W white beads. Note that it has to be symmetrical, since a necklace with 4 black and 7 white beads will have the same number of possibilities as a necklace with 7 black and 4 white beads.

$W \backslash B$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	2	2	3	3	4	4	5	5	6	6	7
3	1	1	2	3	4	5	7	8	10	12	14	16	19
4	1	1	3	4	8	10	16	20	29	35	47	56	72
5	1	1	3	5	10	16	26	38	57	79	111	147	196
6	1	1	4	7	16	26	50	76	126	185	280	392	561
7	1	1	4	8	20	38	76	133	232	375	600	912	1368
8	1	1	5	10	29	57	126	232	440	750	1282	2052	3260
9	1	1	5	12	35	79	185	375	750	1387	2494	4262	7105
10	1	1	6	14	47	111	280	600	1282	2494	4752	8524	14938
11	1	1	6	16	56	147	392	912	2052	4262	8524	16159	29624
12	1	1	7	19	72	196	561	1368	3260	7105	14938	29624	56822

Exercise 10.3: If we include the case where $n = 0$, we obtain the Fibonacci numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

After the first two, each Fibonacci number is the sum of the two previous numbers. In fact, that information may help to make it clear why counting dominoes yields the same series of numbers.

Exercise 10.4: Place numbers on the intersections of the streets to indicate the number of shortest paths that exist to arrive at that point. Starting from a corner of a grid of streets, there is 1 way to get to the starting point: just stand there.

If you wish to arrive at a point that is purely east or purely south, there is again, only one path: go due east or due south for the appropriate number of blocks.

To calculate the number of paths anywhere else in the grid, remember that the final block was either a movement south or a movement east. The number on the grid point just to the north of you tells how many different ways you could have gotten there, and similarly, the number on the point just to the west tells the number of ways you could have gotten there. Add those two numbers to obtain the path count to the new point.

This is *exactly* how Pascal's triangle is constructed, so the numbers on your grid will be exactly the same as the numbers in Pascal's triangle. Refer back to Section 5 for more information.