Kenken For Teachers

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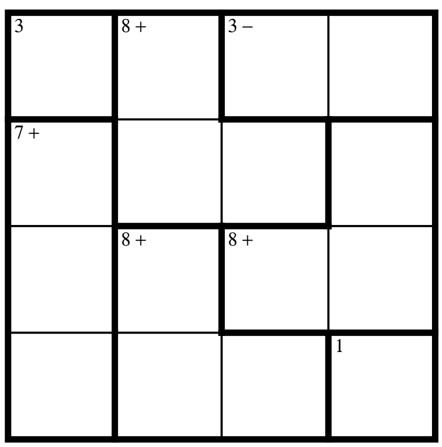
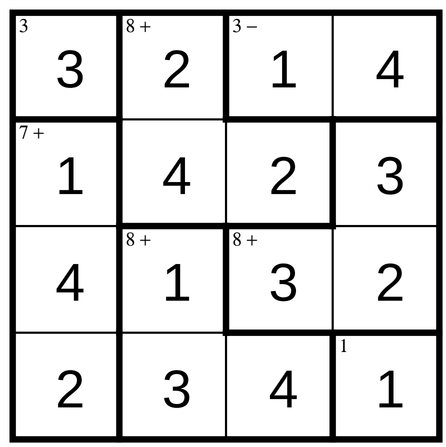
Abstract

Kenken is a puzzle whose solution requires a combination of logic and simple arithmetic skills. The puzzles range in difficulty from very simple to incredibly difficult. Students who get hooked on the puzzle will be forced to drill their simple addition, subtraction, multiplication and division facts.

1 Kenken Rules

On the left in the figure below is a simple Kenken puzzle and on the right is the solved version of that puzzle.





Kenken puzzles are square (this one happens to be 4×4, but they can be of any size). Like Sudoku, the solution requires that an $n \times n$ puzzle contain the digits 1 through n exactly once in each row and column, but in any order. In addition, the puzzle board is divided into possibly irregular "cages", each with an indication of a goal and an operation[1]. If there is no operation indicated in a 1×1 cage, that number is simply to be inserted into the cage. If the operation is $+$ (or $\times$), then the sum (or product) of all the numbers in that cage have to yield the goal number. If the operation is $-$ (or $\div$) then the cage must consist of two squares and the difference (or quotient) of the two numbers must yield the goal number. The numbers can be in any order in the cages.

Check to see that the board on the right above is a solution to the puzzle on the left.

A well-constructed Kenken puzzle, of course, has a unique solution (and all the puzzles in this document are well-constructed).

This document contains a number of puzzles suitable for classroom use in the sections at the end, but there are lots of websites that contain them as well. The author's favorite, at this time, is:

`http://www.nytimes.com/ref/crosswords/kenken.html`

[1] Some advanced puzzles do not even include the operation, but in this paper we will always include it.

where a new set of puzzles of differing size and difficulty is posted every day.

The following Kenken puzzles are not at all easy, but they are beautiful:

`http://www.stanford.edu/~tsnyder/kenken.htm`

2 Classroom Use

Kids can learn a lot just trying to solve a few simple puzzles either by themselves or better, in small groups knowing nothing other than the goal. Have the kids work on an easy puzzle, and then have members of each group explain the sorts of logic they used to solve the puzzle. Make a list on the board of the strategies (even the easiest ones, like "If you know all but one of the numbers in a row/column, then the last number is the one that's missing from the set $\{1, 2, \cdots, n\}$.", where n will probably be 4 for the first puzzles you give them).

In Section 10 is a page of six 4×4 puzzles that are fairly easy. The solutions, in case you need them, are in Section 16.

All the other puzzles are 6×6 (and one very easy 9×9). Often students are much more comfortable with addition/subtraction than multiplication/division, and there is an entire page of 6×6 puzzles that use addition and subtraction goals only.

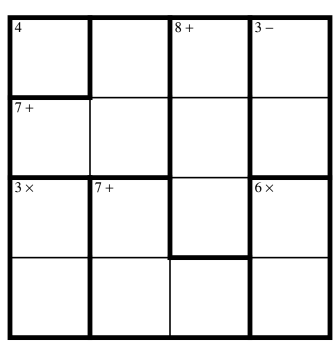
The author has found it useful in presentations to make a transparency of the puzzles to be presented and to project them onto a whiteboard so that the solution can be filled in and discussed using normal whiteboard markers.

For advanced kids, after some practice with puzzle solutions, you can have them try to generate their own puzzles. This is discussed in Section 9.

You can use Kenken puzzles as a gentle introduction to proof. If you look at the descriptions of he solutions to the puzzles in Sections 3, 4 or 8, you will see that each is basically a mathematical proof that the solution is correct. The statements involve possible numbers to fill in the squares, and each requires a reason. You could assign puzzles (probably of size 4×4 to keep the number of steps in the proof manageable) and require a step-by-step explanation of why each conclusion is reached. Students who are afraid of "mathematical proofs" may not even realize that the explanations in the sections referenced above are themselves basically mathematical proofs.

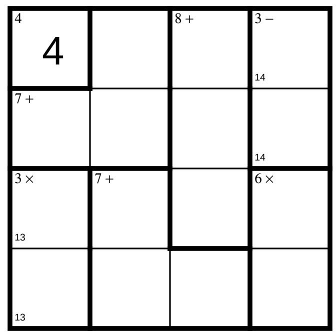
3 Solution Strategies

What follows is a description of some of the techniques that can be used to solve typical Kenken puzzles. We'll begin with a fairly easy 4×4 puzzle, and go through the solution step by step. Here is the puzzle:

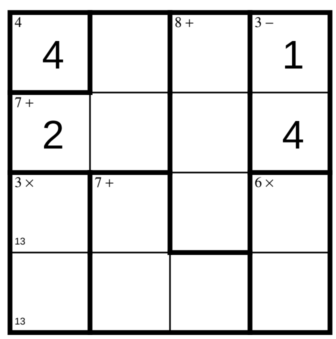


An experienced Kenken solver will probably do most of the steps below in his/her head, but the explanation below is done in complete detail.

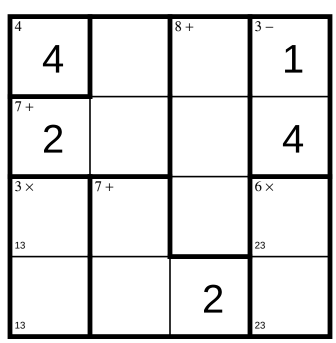
First of all, we can fill in the cages that contain only one square. In this example, the only cage like that is the one in the upper-left corner. In addition, since we know that the numbers in any row and column are chosen from the set $\{1, 2, 3, 4\}$, we can see that the only pairs of numbers that can multiply to yield 3 are 1 and 3, and the only pair of numbers whose difference is 3 are 1 and 4. This constrains the cages in the lower left and upper right to contain those pairs of numbers, but we do not know the order, so we will just indicate with tiny numbers in the squares what the possibilities are. After doing that, our puzzle looks like:



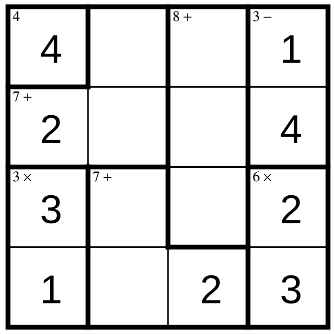
Since there is already a 4 in the first row, 4 is not a possibility in the upper-right square, so it must be 1, and the square below it must therefore contain a 4. In the first column, 1 and 3 must be in the lower left cage and 4 is at the top, so a 2 is the only remaining possibility in the second square down:



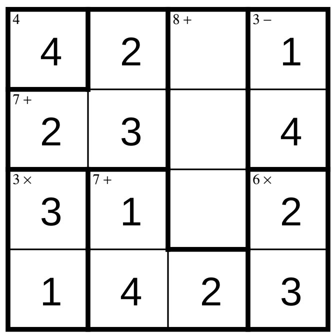
Since 1 and 4 are used in the last column, the two numbers that multiply to yield 6 must be 2 and 3. In the third column, the top cage adds to 8 and since all four numbers in the column add to $1 + 2 + 3 + 4 = 10$, the square at the bottom of that column must contain a 2:



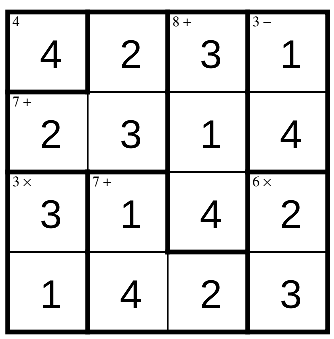
The 2 at the bottom of the third column forces a 3 in the lower right square (and hence a 2 above it). Then that 3 in the lower right forces a 1 in the lower left (and hence a 3 above it):



Now it's easy to make progress: a 4 must go at the bottom of the second column since the other numbers are in use, and above it, a 1, so that the three numbers add to 7. With a 4 and 1 in that second column, the remaining numbers in the column are 2 and 3 and since there's a 2 in the second row already, their order is determined:

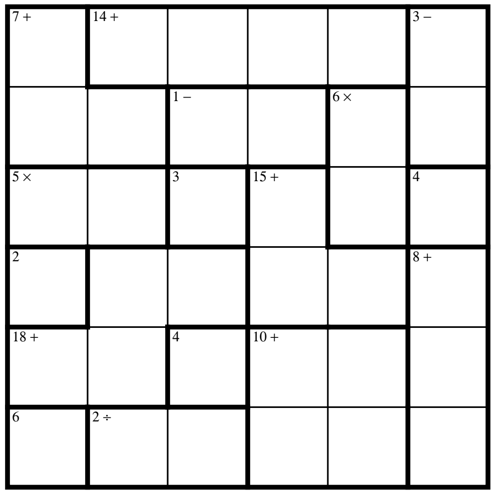


The remaining three empty squares are easy to fill in because we know the other three numbers in each row, and we have the complete solution:

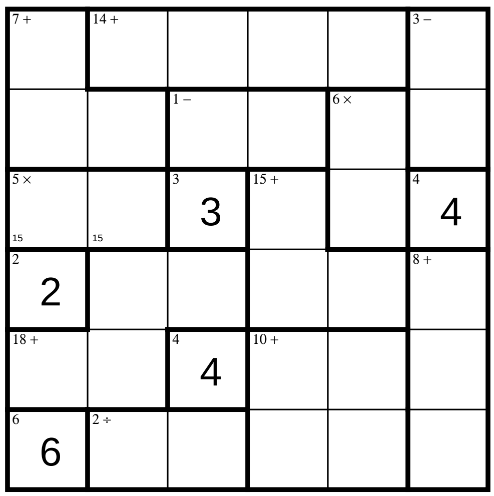


4 An Intermediate Example

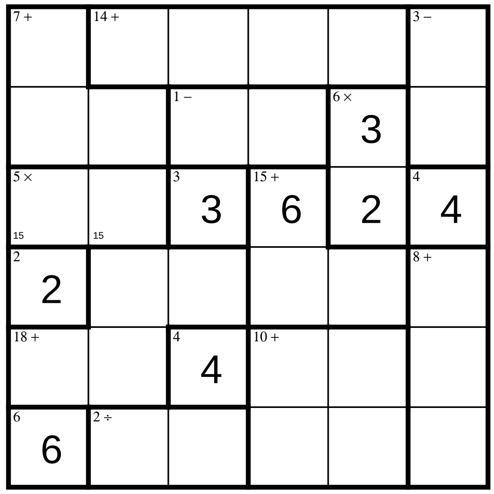
Consider the following 6×6 Kenken puzzle. You may want to try it yourself before reading ahead for the solution. When we say "the solution", we really mean "a solution", since for most puzzles, there are different paths that can lead to the answer.



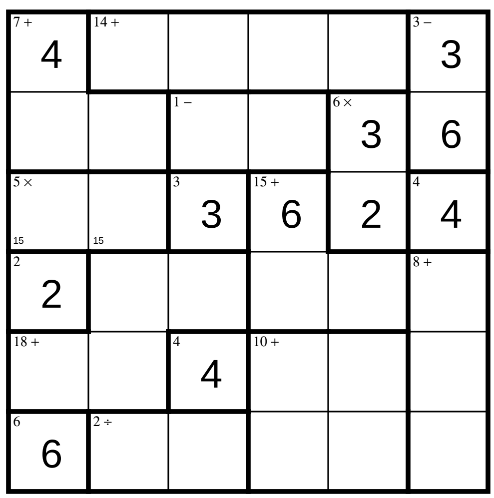
We'll begin as before, filling in the single-square cages and in addition, we can indicate that 1 and 5 are the only candidates in the cage with the $5\times$ goal:



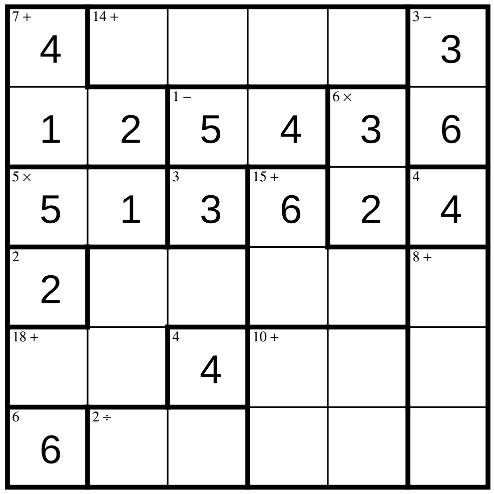
In the third row, we've identified the squares where $1, 3, 4$ and 5 go, so 2 and 6 go in the others. If the 2 were to go in the $15+$ cage, the other two squares would have to add to 13 and that's impossible, since 6 is the largest number we can use. Thus the 6 goes in the $15+$ cage and the 2 in the other (the $6\times$ cage). We can also fill in the other entry (3) in the $6\times$ cage:



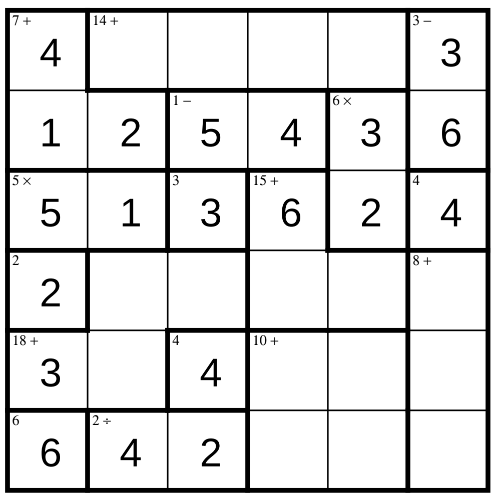
The next step requires a tiny bit of algebra. The bottom four squares in the sixth column add to $4+8=12$, and since all six numbers in any row or column add to $1+2+3+4+5+6=21$, the two entries in the upper-right cage add to $21-12=9$. Now we need two numbers that add to 9 and whose difference is 3 and the only possibilities are 3 and 6. Since there's already a 3 in the second row, the positions of the 3 and 6 are determined. We can use the same trick again in the top row: the 3 in the upper-right and the 14 in the middle four squares add to 17, so the remaining number (in the upper-left corner) must be $21-17=4$:



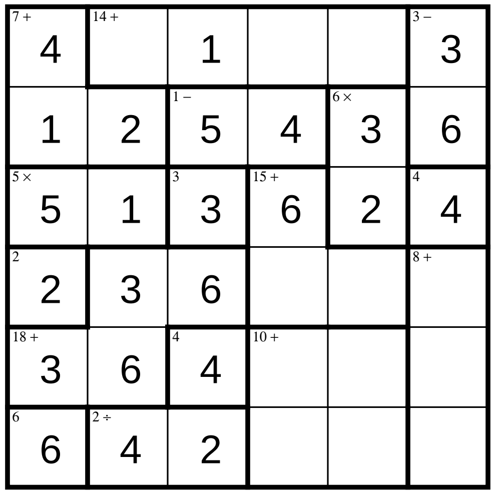
Next, we see that the first and second squares in the second row have to add to 3 to make the cage sum 7 so they must be 1 and 2. The 2 can't go in the first column, so the second row, first column is a 1. That forces a 5 below it and a 1 to the right of that. Finally, the $1-$ cage in the second row must contain a 4 and 5 in some order, since those are the only missing numbers, but the 4 already in column 3 determines the order of those two numbers:



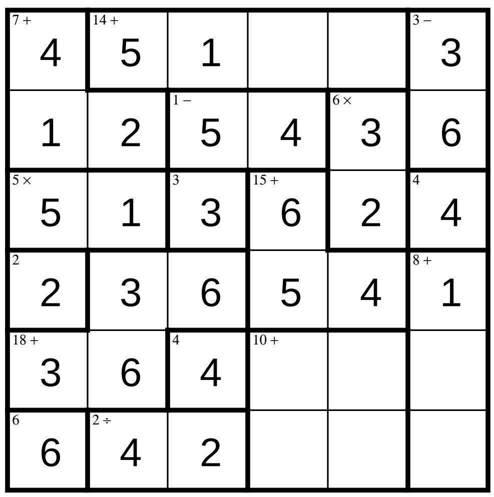
It is easy to fill in the final 3 in the first column, and next we will examine the $2\div$ cage in the bottom row. The only pairs of numbers that can divide to yield 2 are $\{1, 2\}$, $\{2, 4\}$ and $\{3, 6\}$. The last is impossible because of the 6 in the lower-left corner, and it can't be the first, since either the 1 or the 2 would collide with the 1 and 2 that are already in that column. So it must be the $\{2, 4\}$ pair, with the 4 in the second column and the 2 in the third:



Now examine the third column. There's a missing 1 and 6, but if the 1 is in the $18+$ cage, the other two squares will need to add to 14, which is impossible. With the 6 in the $18+$ cage, the other two squares need to add to 9, so they could be $\{3, 6\}$ or $\{4, 5\}$. It can't be $\{4, 5\}$ since that would collide with the 4 at the bottom of the second column, and there is only one way to fill in the 3 and 6:



From here on it’s pretty easy: the top of the second column must be 5, the only two numbers that can go in the $15+$ cage to make the remaining 9 are 4 and 5 and their order is determined. The final number in row four must be 1:



The rest of the puzzle can be solved simply by looking at possible numbers that can go in each square that are not eliminated by other numbers in the same row or column. The solution is:

7+ 4	14+ 5	1	2	6	3− 3
1	2	1− 5	4	6× 3	6
5× 5	1	3 3	15+ 6	2	4 4
2 2	3	6	5	4	8+ 1
18+ 3	6	4 4	10+ 1	5	2
6 6	2÷ 4	2	3	1	5

5 List of Strategies

After working a few puzzles, we can now look at a list of strategies that can be useful. This list is by no means complete. We'll assume here that we are working on 6×6 puzzles, so the obvious modifications must be made for larger or smaller puzzles. Some strategies are obvious, and some not so obvious.

1. If you know 5 of the 6 entries in a row or column, the remaining one can be determined by elimination.

2. The strategy above is simply a special case of the fact that if certain candidates are restricted to particular squares in a row or column, the other squares must contain the other candidates.

3. Just by thinking carefully about the goal for a particular cage, it is often possible to drastically reduce the number of candidates. For example, if there are two squares in a cage and the product is 10, the two squares must contain a 2 and a 5. If a cage consists of three squares in a row and the product is 10, the three entries must be 1, 2 and 5.

4. Each known position of a particular candidate eliminates the possibility of that candidate in all other squares in the row and column, so if you know the positions of a bunch of squares that contain, say, 3, this will severely restrict additional possible positions for 3.

5. Look for cages with restricted goals. For example, a two-square cage with goal $5-$ must contain a 1 and a 6, in some order. If the goal is $4-$ there are two sets of possibilities, if it's $3-$ there are three sets of possibilities, and so on. Thus a cell with a $5-$ goal is more likely to lead to progress than one with $4-$, $3-$, $2-$ or $1-$. Similarly, two-cell cages with a sum of 3 or sum of 11 are good choices, et cetera.

6. It is often easy to bound the size of the entries in a cage. For example, if you have a cage that consists of three squares in a row and the goal is $13+$, then one of the squares must contain a 6. If not, the largest sum you could obtain would be $5 + 4 + 3 = 12$. This cage cannot contain a 1, either, since otherwise the other two entries would have to be 6. If, however, that cage consisted of three squares in an "L" shape, it could contain $\{5, 3, 5\}$ with the 3 in the corner square. This bent cage could also contain a 1, with a $\{6, 1, 6\}$ configuration having the 1 in the corner.

7. Although puzzles with a lot of multiplication and division goals are more difficult for kids because their multiplication facts are generally weaker than their addition facts, a great deal of progress can be made by looking at the multiplication cells whose goal is a multiple of 5. (This applies to

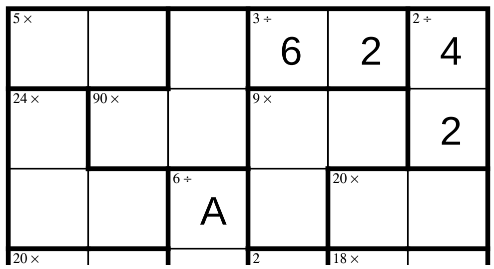
relatively large (for the puzzle size) prime numbers, so to 5 and 7 in 8×8 puzzles, et cetera.) In Section 12, the last three puzzles use only multiplication and division goals. Also, there is an easy 9×9 puzzle of the same form in Section 21.

8. For cages whose goal is a product, it is always a good idea to factor the goal number and to determine from that what possibilities of numbers will multiply to yield that factor.

9. If you have two squares in a row or column and both of them can contain exactly the same two candidates, then those two candidates must go in those two squares, in some order. The same thing applies if you know three candidates that must be within three squares, et cetera.

10. The sum of the entries in any row or column must be equal to $1+2+3+4+5+6 = 21$. Similarly, the product of all the entries in any row must be $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$.

11. The strategy above extends to multiple rows or columns: the sum (or product) of the entries in two rows or columns must be $2 \times 21 = 42$ (or $720^2 = 518400$) and the sum/product of the entries in three rows or columns is $3 \times 21 = 63/720^3 = 373248000$, et cetera. Working with numbers like $720^3 = 37323800$ seems like it would be very difficult, but there's usually no need to do the actual multiplication. As an example, suppose that you have three rows that look like this:



Since you know the numbers in the cages with division as a goal and you know the products of all the other cages, and the cages in the top three rows fill all the squares except for the one in the third row, third column, we can work out the number that goes in that position. Suppose the unknown value in that third row, third column is called A. The products in the two division goal cages, if they were products, would be $6 \times 2 = 12$ and $4 \times 2 = 8$. If we multiply those and all the other product numbers have to satisfy the following equation:

$$5 \times 24 \times 90 \times 9 \times 20 \times 12 \times 8 \times A = 720^3 = 373248000.$$

We could multiply this out, but it's easier to note that $720 = 1 \times 2 \times 3 \times 4 \times 5 \times 6$ and to factor the other numbers in the equation above to convert it to this:

$$5 \times (6 \times 4) \times (3^2 \times 5 \times 2) \times (3^2) \times (5 \times 4) \times (6 \times 2) \times (4 \times 2) \times A = 2^3 \times 3^3 \times 4^3 \times 5^3 \times 6^3.$$

Combining terms, we obtain:

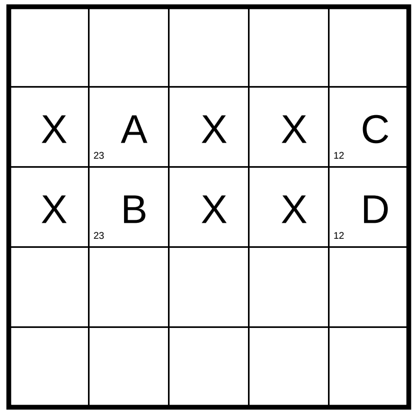
$$6^2 \times 5^3 \times 4^3 \times 3^4 \times 2^3 \times A = 2^3 \times 3^3 \times 4^3 \times 5^3 \times 6^3.$$

Massive cancellation yields:

$$3 \times A = 6,$$

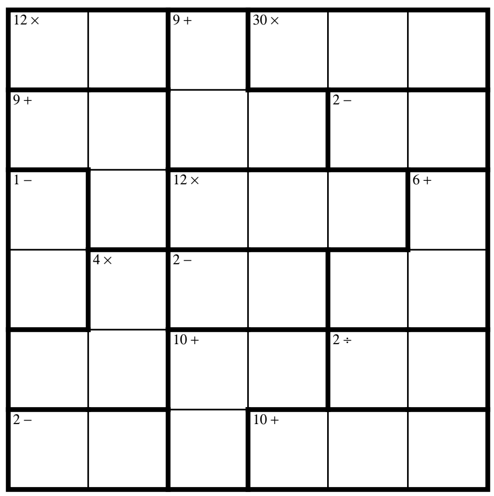
or $A = 2$. We can thus avoid multiplication by doing factoring and cancellation.

12. Suppose a candidate can only be in two squares in a row (or column), and the same candidate can only be in the same two squares of a different row (or column). Then that candidate is eliminated from any other positions in the column (or row) containing either of the two squares. This is illustrated in the example below (which happens to be in a 5×5 puzzle, but the puzzle size is irrelevant).



There must be a 2 in either square A or B and there must also be a 2 in square C or D. The two 2's must therefore fall in either A and D or in B and C. In either case, it is impossible for a 2 to appear in any of the squares marked with an X. (This is called the "X-wing" strategy, as it is also called in Sudoku.)

13. Sometimes you can use parity (odd-even) arguments. Consider the bottom row of the puzzle below:



The difference between the two elements in the leftmost cage is 2, so they must either both be odd or both be even (a mathematician would say that they have the same parity). If they are both odd, then there remains only one odd number, and that cannot go into the rightmost cage (the $10+$ cage), since one odd number and two even numbers must add to yield an odd number and 10 is even. Thus the other odd number must go in the remaining spot (in the other $10+$ cage), so the $10+$ cage at the bottom right of the puzzle contains three even numbers. Those numbers have to be 2, 4 and 6, whose sum is 12, which violates the cage goals. This means that the two numbers in the $2-$ cage on the bottom left must be either 2 and 4 or 4 and 6. Furthermore, there must be two odd numbers in the bottom-right cage (so it must contain either $\{1, 3, 6\}$ or $\{2, 3, 5\}$.

6 Enumeration Exercises

After the class has worked a few puzzles, here are some exercises that they can perform that will help them reduce the number of possibilities in future puzzles. To do it correctly requires patience, which is a great ability when doing problem solving. The exercises here are inspired by the strategy mentioned in Item 3 in Section 5. For all the exercises below, assume that we are working to solve a 6×6 Kenken puzzle.

The solutions appear in Section 7.

1. If a cage consists of three squares in a row, and the goal is $6+$, what are the possibilities?

2. Same as above, but the goal is $7+$?

3. If the cage contains two squares in a row, what are the possibilities with the following three goals: $5-$, $4-$ and $3-$?

4. If a cage contains three squares in an "L" shape, and the goal is $36\times$, what are the possibilities?

5. If a cage contains three squares in an "L" shape, and the goal is $50\times$, what are the possibilities?

6. For cages that are made of a straight line of k squares, how many ways are there the obtain a total of m? If you make a complete table, the values are quite symmetric. Why is that?

7. What is the maximum sum that can be obtained in a three square cage where the squares are arranged in an "L" shape? What's the maximum sum that can be obtained in a four-square cage of any shape?

8. If a four-square cage has $120\times$ as its goal, what are the possible number combinations it can contain?

9. In an 8×8 puzzle, find all the ways to fill a four-square cage whose goal is $192\times$.

10. This is a *much* harder problem for advanced students. If a six-square cage in a 12×12 Kenken puzzle has a goal of 6480, what are the possibilities? Hint: there are 37 solutions. If you can't find all of them, find as many as you can.

7 Enumeration Solutions

1. If a cage consists of three squares in a row, and the goal is $6+$, what are the possibilities?

   Solution: The three numbers must be 1, 2 and 3 in some order. There are 6 possible orders: 123, 132, 213, 231, 312 and 321.

2. Same as above, but the goal is $7+$?

   Solution: The three numbers must be 1, 2 and 4 in some order. Again, there are 6 possible orderings of those numbers.

3. If the cage contains two squares in a row, what are the possibilities with the following three goals: $5-$, $4-$ and $3-$?

   Solution: For the $5-$ goal, the only possibilities are $\{1, 6\}$. For $4-$, the possibilities are either $\{1, 5\}$ or $\{2, 6\}$. For the goal $3-$, the possibilities are either $\{1, 4\}$, $\{2, 5\}$ or $\{3, 6\}$.

4. If a cage contains three squares in an "L" shape, and the goal is $36\times$, what are the possibilities?

   Solution: The numbers could be $\{2, 3, 6\}$ and they can appear in the cage in any positions. If the numbers are $\{1, 6, 6\}$ or $\{3, 3, 4\}$ then the non-duplicated number must be in the corner position.

5. If a cage contains three squares in an "L" shape, and the goal is $50\times$, what are the possibilities?

   Solution: The corner square must contain a 2 and there are 5's in both the other squares.

6. For cages that are made of a straight line of k squares, how many ways are there the obtain a total of m? If you make a complete table, the values are quite symmetric. Why is that?

   Solution: In the table below are listed all the results for straight-line cages consisting of 1 through 6 squares. Find the number of squares on the top, and the desired total on the left, and the number of combinations that will yield that total appears in the table. For example, there are 3 ways to obtain a total of 14 using a four-square straight-line cage. Here they are: $\{1, 2, 5, 6\}$, $\{1, 3, 4, 6\}$ and $\{2, 3, 4, 5\}$.

Total	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	0	0	0	0	0
3	1	1	0	0	0	0
4	1	1	0	0	0	0
5	1	2	0	0	0	0
6	1	2	1	0	0	0
7	0	3	1	0	0	0
8	0	2	2	0	0	0
9	0	2	3	0	0	0
10	0	1	3	1	0	0
11	0	1	3	1	0	0
12	0	0	3	2	0	0
13	0	0	2	2	0	0
14	0	0	1	3	0	0
15	0	0	1	2	1	0
16	0	0	0	2	1	0
17	0	0	0	1	1	0
18	0	0	0	1	1	0
19	0	0	0	0	1	0
20	0	0	0	0	1	0
21	0	0	0	0	0	1

   The values in the table form a symmetric pattern, since if we replace every value x by $6 - x$, we will also obtain equal sums.

7. What is the maximum sum that can be obtained in a three square cage where the squares are arranged in an "L" shape? What's the maximum sum that can be obtained in a four-square cage of any shape?

   Solution: Since the "L" cell sits on two rows and two columns, you can put a 5 in the corner and two 6's on the ends for a total of 17. Four cells arranged in a square or in a zig-zag pattern can contain two 5's and two 6's, for a total of 22.

8. If a four-square cage has $120\times$ as its goal, what are the possible number combinations it can contain?

   Solution: The easiest way to do problems like this is first to factor the number: $120 = 2^3 \cdot 3 \cdot 5$, so it has five prime factors. If you multiply 5 by anything, it will be too large, so one of the four squares has to contain a 5. The others will contain some combination of 1's (don't forget to consider 1 as a factor!), 2's, 3's, 4's and 6's which are all possible from the remaining factors. An organized way to find all the possibilities for the three squares that do not contain a 5 is to list them in numerical order, as: 146, 226, 234. Thus the four cases are $\{1, 4, 5, 6\}$, $\{2, 2, 5, 6\}$ and $\{2, 3, 4, 5\}$. If the four squares in the cage lie on a straight line, then obviously $\{2, 2, 5, 6\}$ is impossible.

9. In an 8×8 puzzle, find all the ways to fill a four-square cage whose goal is $192\times$.

Solution: There are five of them: $\{1,3,8,8\}$, $\{1,4,6,8\}$, $\{2,2,6,8\}$, $\{2,3,4,8\}$ and $\{2,4,4,6\}$. There is one other way to achieve a product of 192 with four numbers chosen between 1 and 8; namely: $\{3,4,4,4\}$, but it is impossible to arrange four squares in a cage that would not have two of the 4's in the same row or column.

10. This is a *much* harder problem for advanced students. If a six-square cage in a 12×12 Kenken puzzle has a goal of 6480, what are the possibilities? Hint: there are 37 solutions. If you can't find all of them, find as many as you can.

    Solution: Here is a list in "numerical" order:

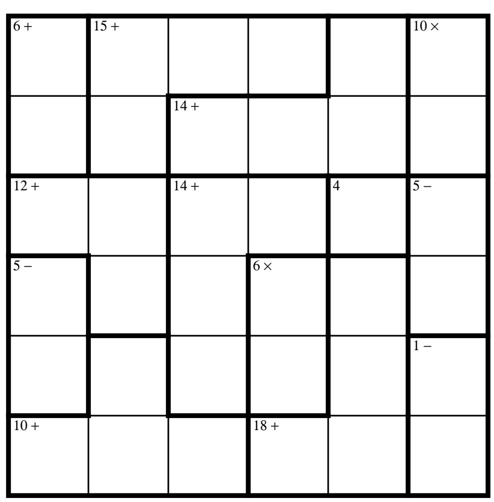
$\{1,1,5,9,12,12\}$	$\{1,1,6,9,10,12\}$	$\{1,1,8,9,9,10\}$	$\{1,2,3,9,10,12\}$
$\{1,2,4,9,9,10\}$	$\{1,2,5,6,9,12\}$	$\{1,2,5,8,9,9\}$	$\{1,2,6,6,9,10\}$
$\{1,3,3,5,12,12\}$	$\{1,3,3,6,10,12\}$	$\{1,3,3,8,9,10\}$	$\{1,3,4,5,9,12\}$
$\{1,3,4,6,9,10\}$	$\{1,3,5,6,6,12\}$	$\{1,3,5,6,8,9\}$	$\{1,3,6,6,6,10\}$
$\{1,4,4,5,9,9\}$	$\{1,4,5,6,6,9\}$	$\{1,5,6,6,6,6\}$*	$\{2,2,2,9,9,10\}$
$\{2,2,3,5,9,12\}$	$\{2,2,3,6,9,10\}$	$\{2,2,4,5,9,9\}$	$\{2,2,5,6,6,9\}$
$\{2,3,3,3,10,12\}$	$\{2,3,3,4,9,10\}$	$\{2,3,3,5,6,12\}$	$\{2,3,3,5,8,9\}$
$\{2,3,3,6,6,10\}$	$\{2,3,4,5,6,9\}$	$\{2,3,5,6,6,6\}$	$\{3,3,3,3,8,10\}$*
$\{3,3,3,4,5,12\}$	$\{3,3,3,4,6,10\}$	$\{3,3,3,5,6,8\}$	$\{3,3,4,4,5,9\}$
$\{3,3,4,5,6,6\}$			

    Note that it is impossible to make a six-square cage that lies on four rows and four columns, so the combinations $\{1,5,6,6,6,6\}$ and $\{3,3,3,3,8,10\}$ are actually impossible.

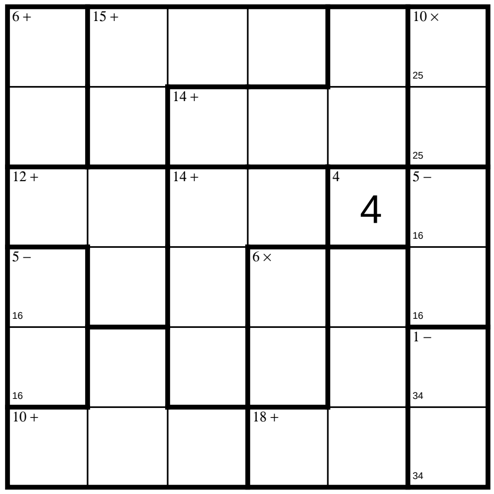
    In actual 6×6 puzzles, the situation is never anywhere near this complex. There are at most 2 possibilities for two-square cages, 3 possibilities for three-square cages and 4 possibilities for four-square cages. In an 8×8 puzzle, those numbers are at most: 2 possibilities for two-square cages, 4 possibilities for three-square cages and 6 possibilities for four-square cages.

8 A Hard Example

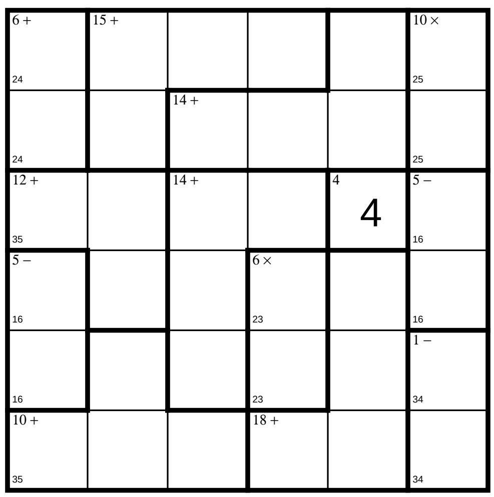
We'll finish up with one more difficult example.



At first it doesn't seem like there is much we can do. Fill in the single-square cage and fill in a few obvious pairs. After we've gotten 2, 5, 1 and 6 in the rightmost column, the other two squares must contain 3 and 4:

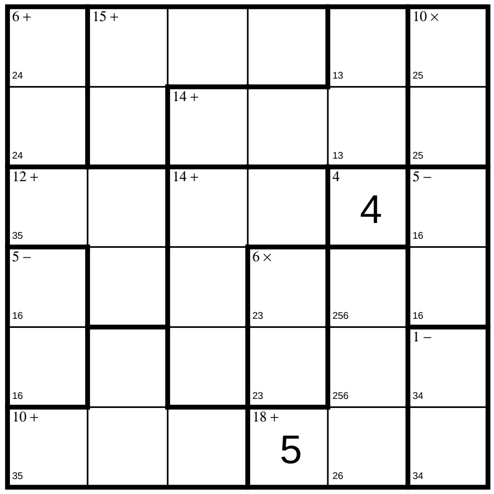


Now in the upper-left cage, since there's already a 1 and 6 in the column, the only other pair of numbers that can add to 6 are 2 and 4. Also note that in the fourth row, we have two squares that both have only $\{1, 6\}$ as possibilities, so it is impossible to have either a 1 or a 6 in the top of the $6\times$ cage on row four, so that $6\times$ cage must contain a 2 and a 3. Finally, in the first column with 1, 2, 4 and 6 occupying known squares, the other two squares must contain 3 and 5, in some order:

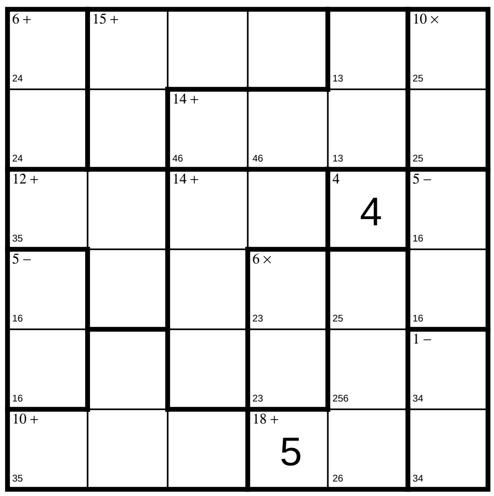


Next, consider the situation in the $18+$ cage near the bottom-right of the puzzle. The three vertical cells must add to at least 12, since the largest number we can put in row six, column four is a 6. The 4 in column five forces us to have either $\{6, 5, 3\}$, $\{6, 5, 2\}$ or $\{6, 5, 1\}$ as the three vertical entries.

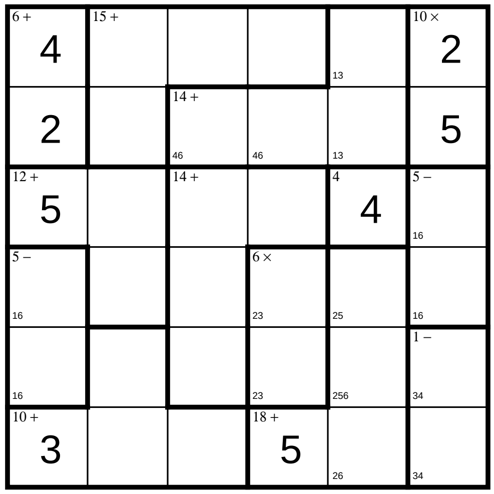
Now we can apply the X-wing strategy (see Item 12 in Section 5). Consider the two vertical cages at the upper-right and upper-left of the puzzle. Both contain a 2, so it is impossible for a 2 to appear in the top two positions of column five, so the 2 must be included in the three vertical squares in the $18+$ cage. Thus the only possibility is $\{6, 5, 2\}$ whose sum is 13 forcing a 5 in row six, column 4, and forcing 1 and 3 in some order in the top two positions of column five. Of course the 5 in the bottom row eliminates the possibility of a 5 in the fifth column, bottom row:



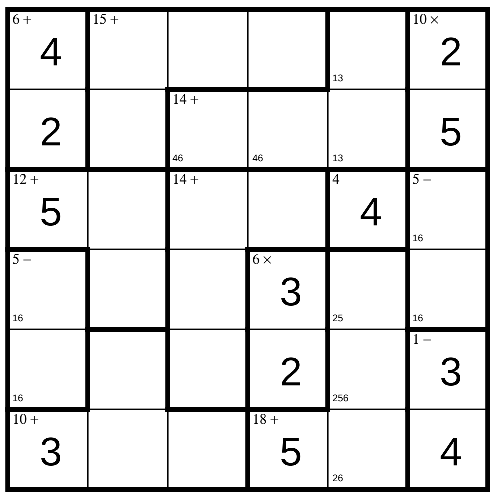
The 1 and 6 as candidates in the first and last positions of row four eliminate the possibility of a 6 in row four, column five. The 1 and 3 in the top two entries of column five force the other two entries in the $14+$ cage to add to 10 and this can only occur with a 4 and a 6:



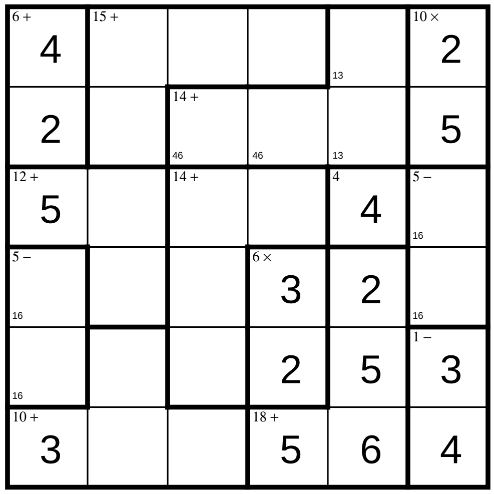
The guaranteed 4 in the $14+$ cage in row two eliminates the possibility of a 4 in row two, column one, and that locks down the entries in the cages in the upper-left and upper-right corners. Another obvious thing that occurs is that the 5 in the bottom row forces a 3 at the bottom of the first column and thus a 5 in the first column of the third row:



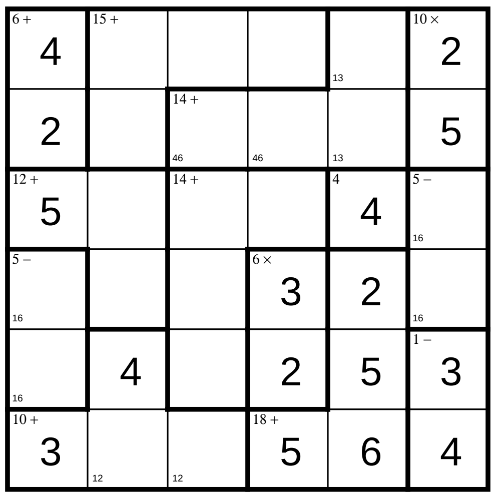
The 3 in the bottom row forces a 4 in the lower-right corner and a 3 above it. That forces the order of the 2 and 3 in the $6\times$ cage:



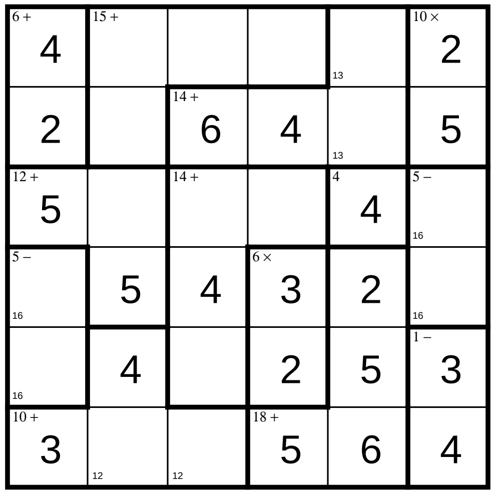
Now consider the bottom row. If there is a 6 in the $10+$ cage the two remaining squares will take the total to at least 11, so 6 must go in column five. It must have a 5 above it, and a 2 above that:



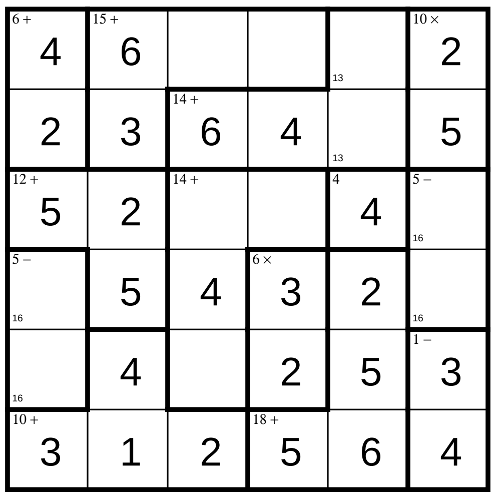
The two open entries in the bottom row must contain a 1 and 2 in some order, so to make a total of 10, the fifth row, second column must have a 4:



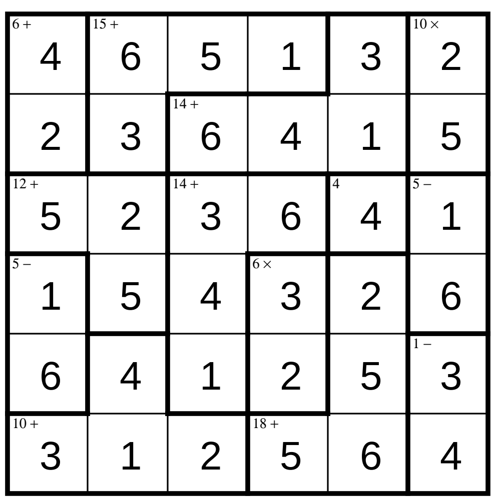
The two unmarked squares in row four must contain a 4 and a 5 in a forced order, and that, in turn, forces the order of the entries in the third and fourth positions of row two:



Now things are easy: we're missing a 2 in the $12+$ slot in row 3 and that forces the two entries in the bottom row. The missing 3 and 4 in column 2 are easy to assign:



From here on things are very easy, and here is the complete solution:



9 Generating Kenken Puzzles

After the students have solved a few puzzles, a very interesting exercise for them is to have teams or two or three generate their own Kenken puzzles and then have the teams exchange those puzzles with other teams who will then attempt to solve them (or to prove that they either can't be solved, or that there are multiple solutions). One way to do this is to have each team draw their puzzle on the board, and then swap the teams around so that students can try to solve the puzzles created by others. This can be fairly time-consuming, and there can be problems with non-uniqueness of the solutions, as well as errors, both by the creators and by the solvers of the puzzles.

As was the case with the initial introduction of Kenken to the class, it's probably best just to have the kids try to make some puzzles, and then, after they've seen what some of the problems are, you can make lists of strategies that are useful for puzzle creation. I recommend that the first puzzles that the teams try to create be 4×4. Be sure to tell the class that it is important that the puzzles created not only have a solution, but that solution be unique.

The author of this article does not claim to be an expert in Kenken puzzle creation (although his software was used to create and check all the examples used here). Thus it may be possible that many interesting ideas are omitted in what follows.

9.1 Generating Latin Squares

An easy way to begin is first to generate the numbers that will provide the solution. The only requirement for this is that each row and column contains all the numbers from 1 to n (for an $n \times n$ puzzle). The cages and operations can then be added to create the final puzzle. Sometimes it may be necessary, during cage and operation selection, to modify the numbers in the grid slightly.

These $n \times n$ grids of numbers with no duplicates in rows or columns are called "Latin squares", and a great deal is known about them. For small grids, like the 4×4 grids that will be used for the first puzzles, they are fairly easy to generate, just by trial and error.

But there are some easy ways to generate such squares totally mechanically in such a way that the resulting squares seem somewhat random. Note that once you have a Latin square, you can make another by rearranging the rows in any way, and then rearranging the columns in any way. Another way to make a new Latin square from an old one is to take any 1-to-1 function mapping $1 \cdots n$ to $1 \cdots n$ and applying that function to all the elements in an existing Latin square. There are 24 rearrangements of the rows, 24 rearrangements of the columns, and 24 suitable mappings, so just using these techniques, a single initial Latin square can have up to $24 \times 24 \times 24 = 13824$ modifications. (A lot of these modifications yield the same Latin square; in fact, of the 4×4 variety, there are only 576 different examples.)

So you can start with any Kenken puzzle, apply some of the transformations above, and get one that looks completely different with almost no effort. Also note that you can start with one that is very regular and modify it as explained above. Here is an easy beginning 4×4 Latin square, and the same pattern can be used to generate a starting Latin square of any size:

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

Things are a little more tricky with larger squares, so the techniques above may be more useful as the size of the puzzle increases. In fact, there are 161280 of the 5×5 squares and 812851200 of the 6×6 squares.

Notice that as you're building the cages in your square you can continue to use some of the transformations above. Suppose that you're generally generating the cages from top to bottom and you can see that it would be better in some way to swap some of the lower rows, then as long as there are no cages with operations and goals defined on those rows, you can just go ahead and swap them.

9.2 Generating Cages and Cage Goals

An easy method to generate a puzzle uses a trial-and-error approach. Divide the Latin square into cages and then figure out operations and goals for those cages. For example, if you've got a cage with three squares that contain a 1, 3 and 4, you could have as a goal either $8+$ or $12\times$. In addition, for two-cage goals there is always the subtraction goal and, if one of the numbers divides the other, a division goal.

If you are trying to make a puzzle that uses division as a goal, make sure that the Latin square you generate has sufficient adjacent squares where one is a multiple of the adjacent one to make as many division slots as you desire.

After you've generated cages and goals, start with the proposed puzzle and try to solve it completely logically. If you can do this, it is a valid puzzle. If you find that you need to make a guess, it may be that there are multiple solutions and the puzzle is therefore not valid.

As you are trying to solve the puzzle, if you do get stuck and have no more logical steps that you can take, consider changing the so far undetermined cages either in terms of shape or goal so that additional steps can be made.

It is easiest at first to generate easy puzzles, and one way to do this is to begin by selecting cage shapes over the known solution numbers such that the cages' contents are completely determined. Here are a few examples (assuming we are constructing a 6×6 puzzle):

1. For two-square cages, the following goals allow only one pair of numbers in the solution: $5-$, $11+$, $10\times$, $10+$, $3+$, $24\times$, $30\times$. (This is not a complete list; try to find others.)
2. If the cage consists of three squares in a row, the following targets allow exactly one set of three numbers: $6+$, $7+$, $15+$, $14+$, $10\times$, $15\times$. Again, this is not a complete list.
3. For a cage with three squares in an L shape, sometimes the goal completely determines not only the values, but the locations for those values. This occurs when two of the numbers are the same. For example, $2\times$, $3\times$, $25\times$, $4+$, $5+$, $75\times$. There are many more.

Note that generally as the number of possible combinations of numbers that can fill a slot increases, the puzzle becomes more difficult. For a two-square cage, the goals of $5-$, $4-$, $3-$, $2-$ and $1-$ have, respectively, 1, 2, 3, 4 and 5 possible combinations of numbers that satisfy the goal.

9.3 Proving Uniqueness of the Solution

In general, this is difficult, and the best way, for puzzles where a guess has to be made, is to use a computer program to check for the uniqueness of the solution. Puzzles that require a guess are not as nice as puzzles that can be solved using pure logic, so it's usually probably best to modify cages if you reach a stage where a guess seems to be required.

9.4 Computer Generation of Puzzles

The author is not an expert at this, but at least the methods he used in his computer program (called "`kengen`") can be described here.

First, `kengen` generates a random Latin square. It then generates cages by randomly picking adjacent squares and combining them into cages with some restrictions. For example, there is a parameter that limits the maximum number of squares in a cage, and if the combination of two adjacent squares makes a cage that is too large, it is not done. Squares are combined in this way at random until a suitable number of cages is formed.

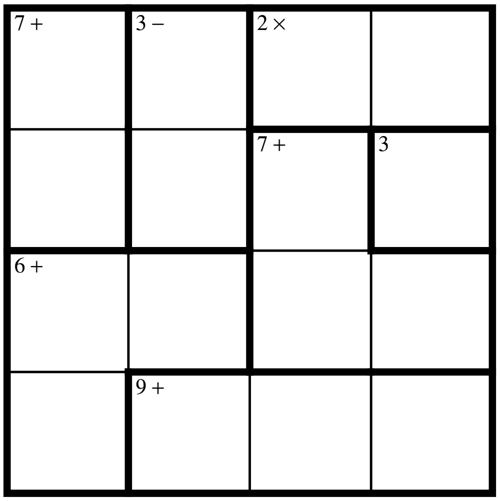
Next, `kengen` assigns possible goals to the cages. If it is trying to make easy puzzles, it looks for easily-satisfied goals, as described above. If not, it just invents goals at random.

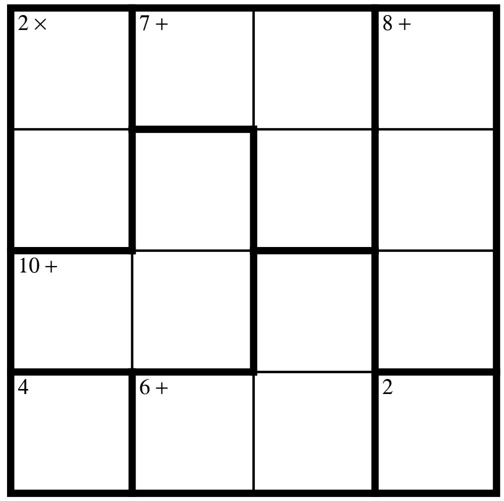
Finally, `kengen` tries to solve the problem, using a number of heuristics described in the sections above dealing with how to solve Kenken puzzles. The heuristics are graded in terms of difficulty, so as the solution is attempted, `kengen` keeps track of how many of the harder techniques are required so it can make an estimate of how hard the puzzle is.

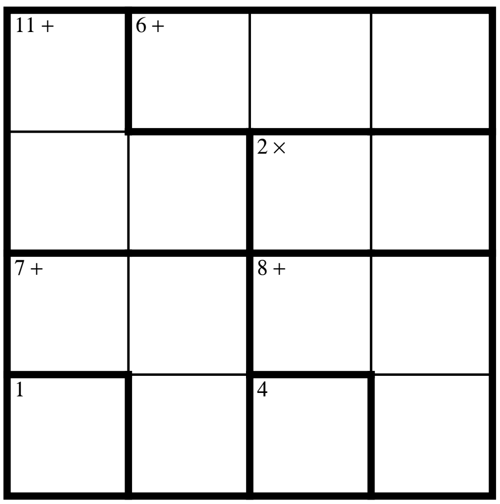
If the puzzle requires a guess, `kengen` either rejects it (if it's looking for relatively easy puzzles), or it makes a guess and continues the solution. If the guess fails, it tries another guess from the same point, et cetera. If it succeeds, it also tries the other possible guesses, and if any of those succeed, the puzzle is bad because it has multiple solutions. Obviously, this guessing may have to be applied recursively.

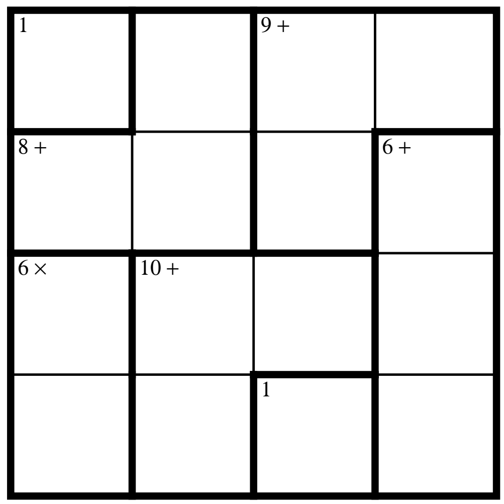
If a puzzle fails for any reason, `kengen`, just makes up a new grid and cages and tries again until it succeeds. Generally, not too many proposed puzzles fail, especially if it is trying to make easy-to-solve puzzles.

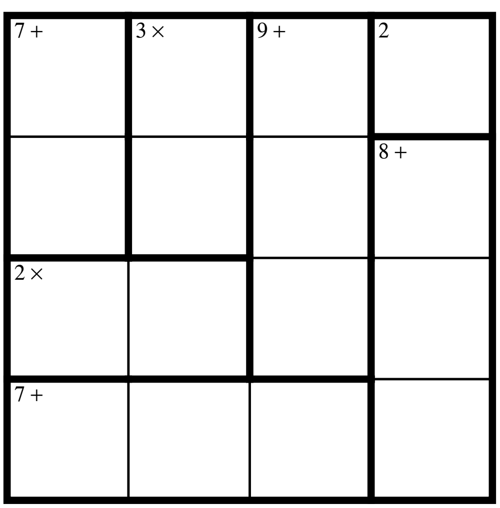
10 Easy Puzzles (Solutions: Section 16)

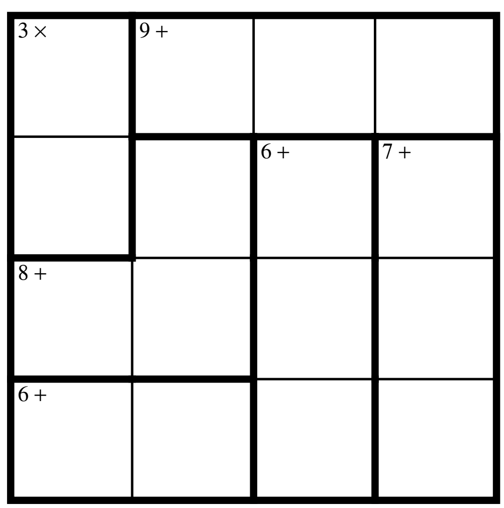




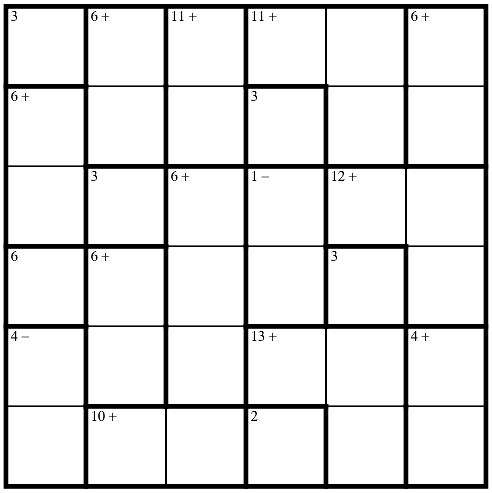


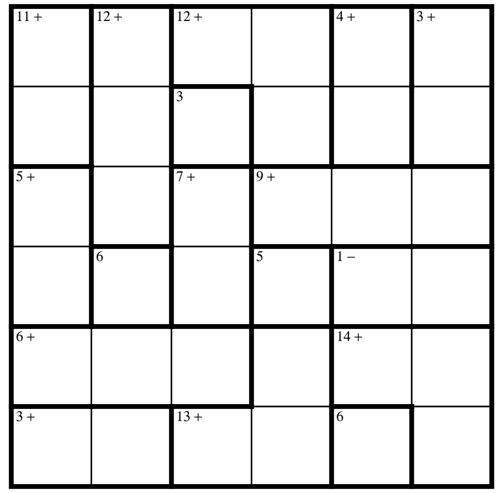


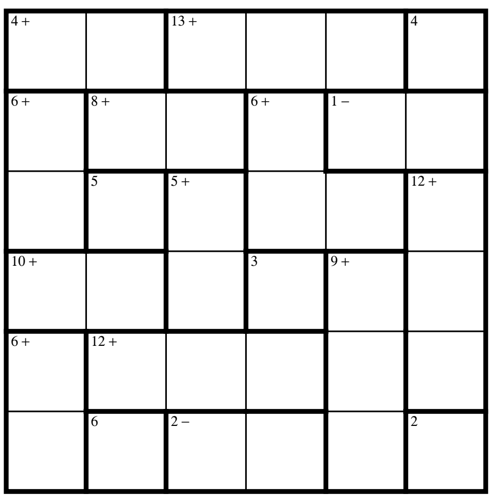


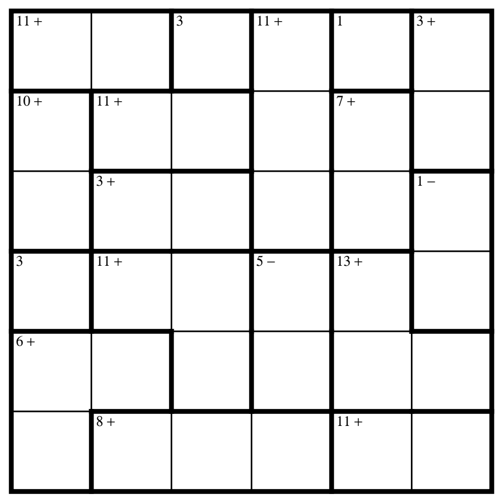


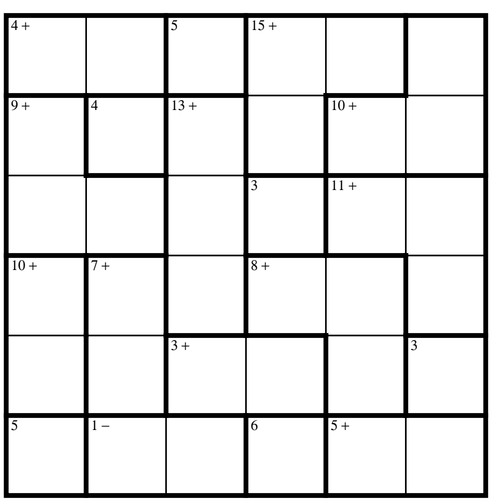
11 Easy Add/Subtract Puzzles (Solutions: Section 17)

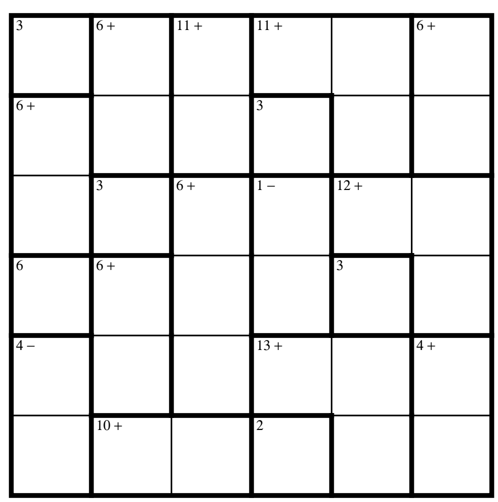




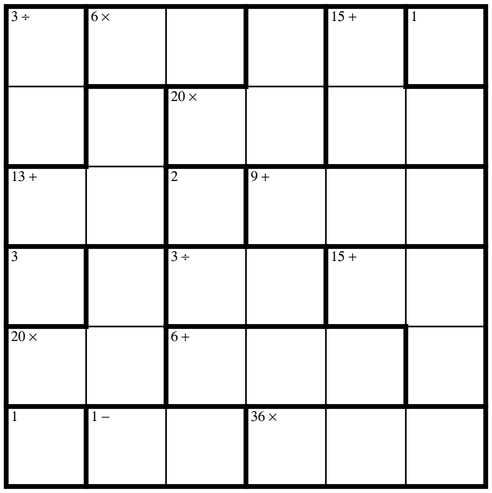


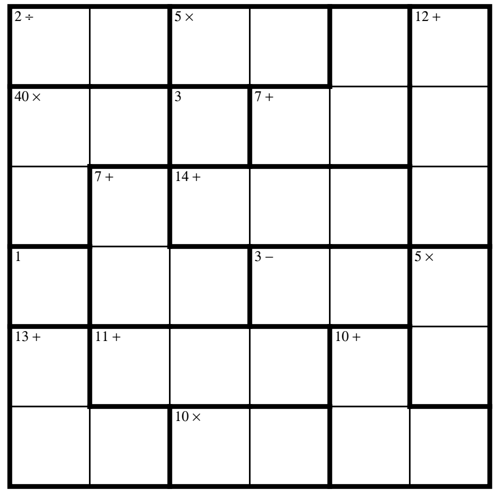


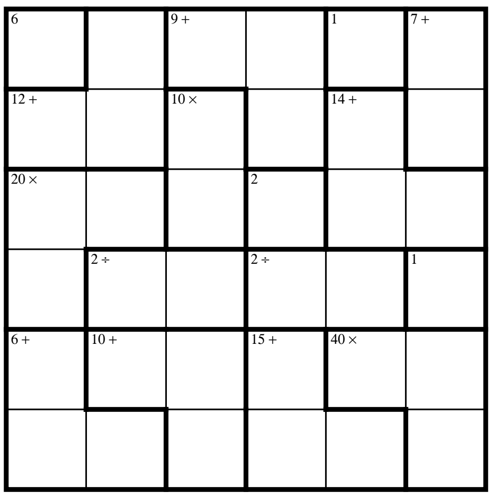


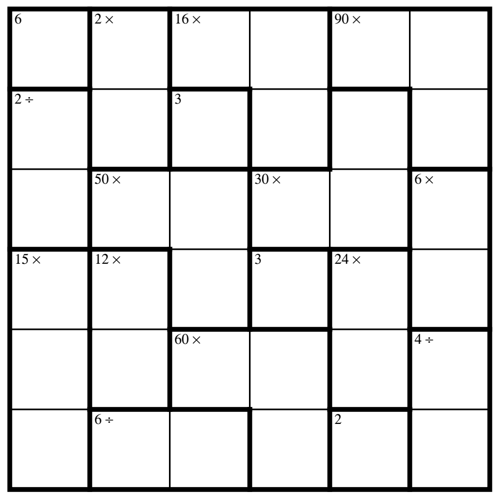


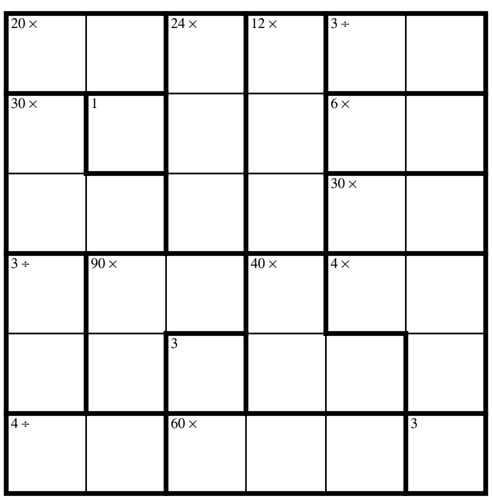
12 Easy 6×6 Puzzles (Solutions: Section 18)

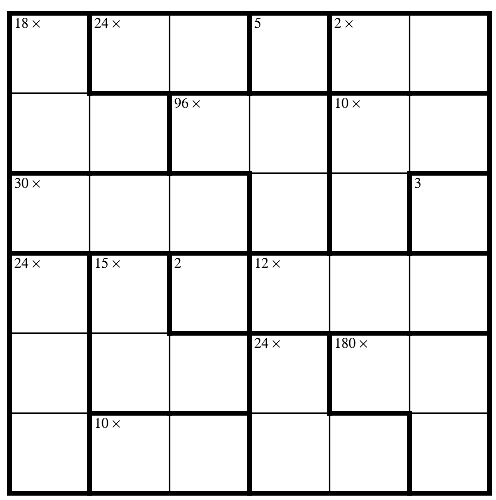




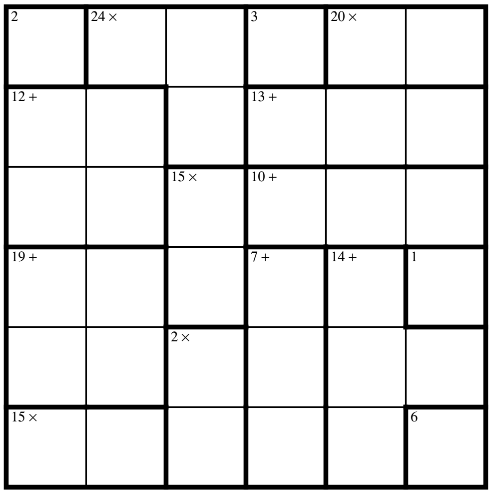


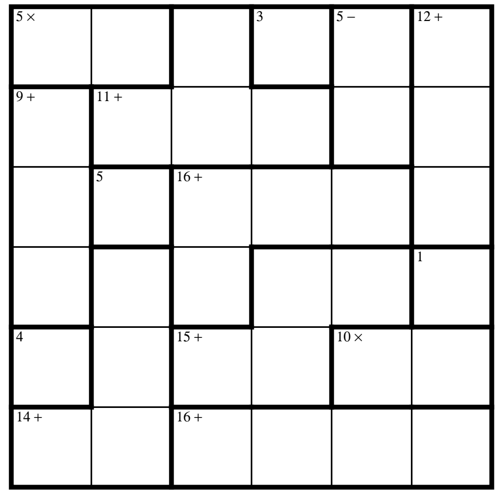


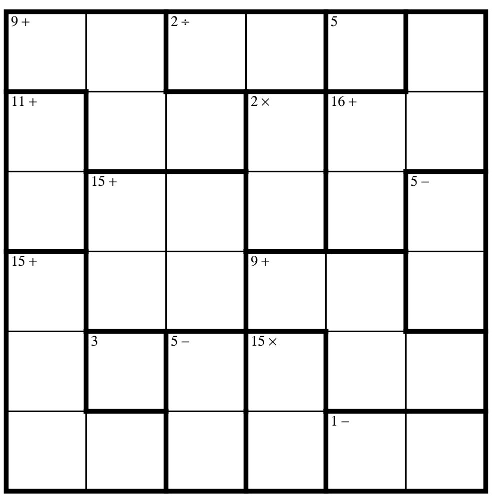


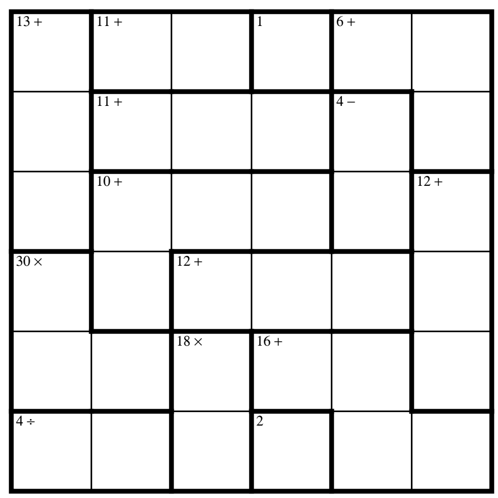


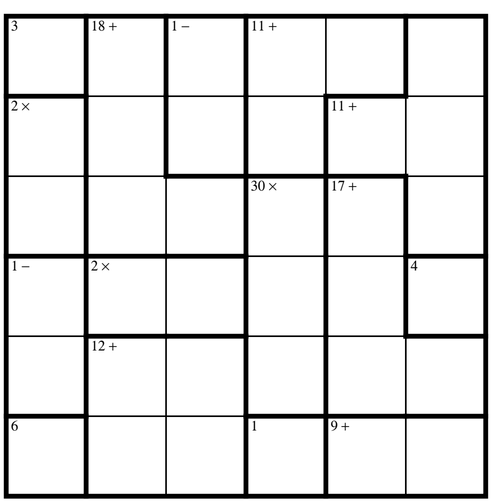
13 Medium 6×6 Puzzles (Solutions: Section 19)

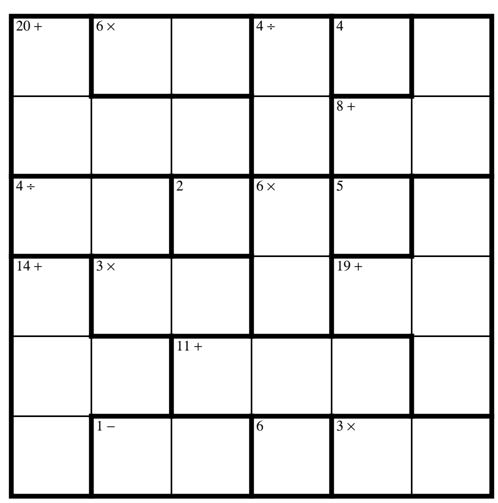




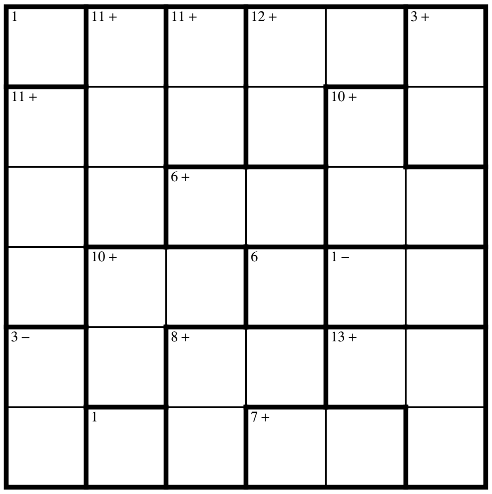


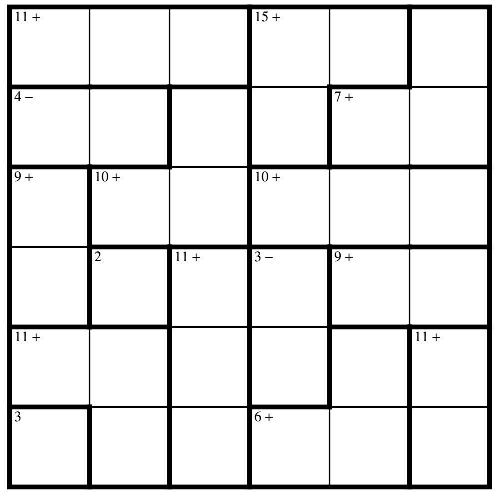


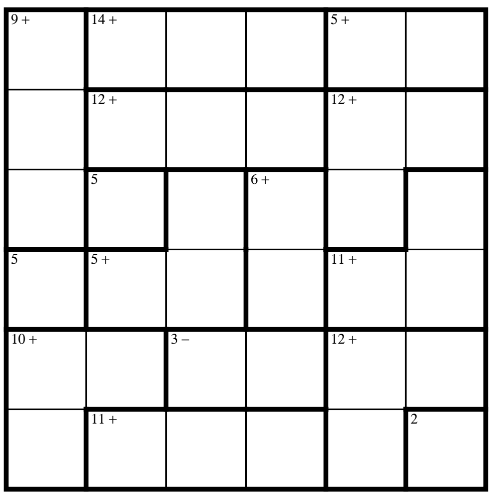


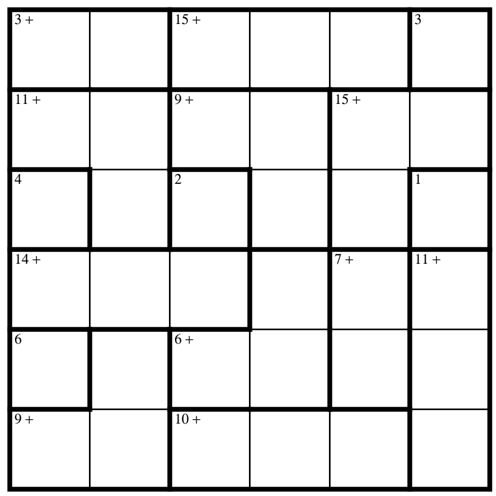


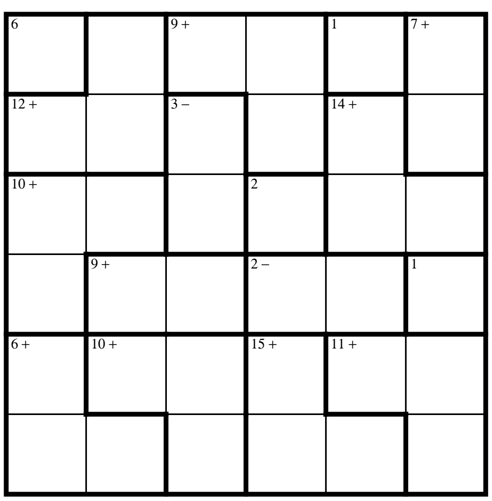
14 Difficult 6×6 Puzzles (Solutions: Section 20)

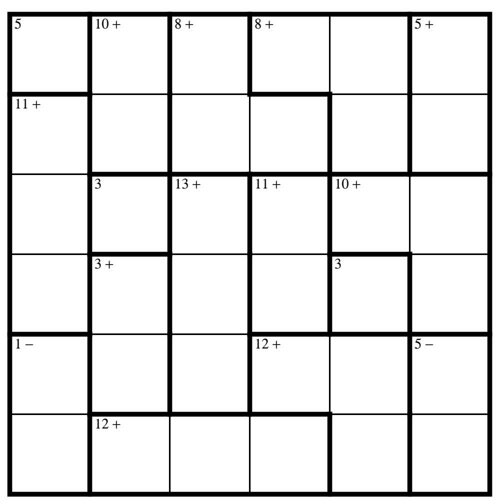












15 Easy 9×9 Puzzle (Solution: Section 21)

48 $\times$	6 $\div$	80 $\times$		42 $\times$	189 $\times$		36 $\times$	
144 $\times$	216 $\times$		15 $\times$		1	70 $\times$		16 $\times$
		1	336 $\times$		45 $\times$	144 $\times$		
147 $\times$		45 $\times$	4				2	
	8			3 $\div$		5 $\times$	36 $\times$	
10 $\times$		144 $\times$		36 $\times$				35 $\times$
10 $\times$			18 $\times$	224 $\times$		24 $\times$	72 $\times$	
	21 $\times$			1				6

16 Easy Solutions

7 + 3	3 − 4	2 × 1	2
4	1	7 + 2	3 3
6 + 2	3	4	1
1	9 + 2	3	4

2 × 1	7 + 2	4	8 + 3
2	3	1	4
10 + 3	4	2	1
4 4	6 + 1	3	2 2

11 + 4	6 + 1	3	2
3	4	2 × 2	1
7 + 2	3	8 + 1	4
1 1	2	4 4	3

1 1	3	9 + 2	4
8 + 4	1	3	6 + 2
6 × 3	10 + 2	4	1
2	4	1 1	3

7 + 3	3 × 1	9 + 4	2 2
4	3	2	8 + 1
2 × 1	2	3	4
7 + 2	4	1	3

3 × 1	9 + 2	4	3
3	1	6 + 2	7 + 4
8 + 4	3	1	2
6 + 2	4	3	1

17 Easy Add/Subtract Solutions

[3] 3	[6 +] 5	[11 +] 6	[11 +] 1	4	[6 +] 2
[6 +] 2	1	5	[3] 3	6	4
4	[3] 3	[6 +] 2	[1 −] 5	[12 +] 1	6
[6] 6	[6 +] 2	1	4	[3] 3	5
[4 −] 5	4	3	[13 +] 6	2	[4 +] 1
1	[10 +] 6	4	[2] 2	5	3

[11 +] 5	[12 +] 4	[12 +] 6	2	[4 +] 3	[3 +] 1
6	5	[3] 3	4	1	2
[5 +] 4	3	[7 +] 5	[9 +] 1	2	6
1	[6] 6	2	[5] 5	[1 −] 4	3
[6 +] 3	2	1	6	[14 +] 5	4
[3 +] 2	1	[13 +] 4	3	[6] 6	5

[4 +] 3	1	[13 +] 2	6	5	[4] 4
[6 +] 5	[8 +] 2	6	[6 +] 1	[1 −] 4	3
1	[5] 5	[5 +] 4	2	3	[12 +] 6
[10 +] 6	4	1	[3] 3	[9 +] 2	5
[6 +] 2	[12 +] 3	5	4	6	1
4	[6] 6	[2 −] 3	5	1	[2] 2

[11 +] 5	6	[3] 3	[11 +] 4	[1] 1	[3 +] 2
[10 +] 4	[11 +] 5	6	2	[7 +] 3	1
6	[3 +] 1	2	5	4	[1 −] 3
[3] 3	[11 +] 2	5	[5 −] 1	[13 +] 6	4
[6 +] 1	3	4	6	2	5
2	[8 +] 4	1	3	[11 +] 5	6

[4 +] 3	1	[5] 5	[15 +] 4	6	2
[9 +] 1	[4] 4	[13 +] 3	5	[10 +] 2	6
2	6	4	[3] 3	[11 +] 5	1
[10 +] 4	[7 +] 2	6	[8 +] 1	3	5
6	5	[3 +] 1	2	4	[3] 3
[5] 5	[1 −] 3	2	[6] 6	[5 +] 1	4

[3] 3	[6 +] 5	[11 +] 6	[11 +] 1	4	[6 +] 2
[6 +] 2	1	5	[3] 3	6	4
4	[3] 3	[6 +] 2	[1 −] 5	[12 +] 1	6
[6] 6	[6 +] 2	1	4	[3] 3	5
[4 −] 5	4	3	[13 +] 6	2	[4 +] 1
1	[10 +] 6	4	[2] 2	5	3

18 Easy 6×6 Solutions

3÷ 6	6× 2	3	4	15+ 5	1 1
2	3	20× 5	1	6	4
13+ 4	6	2 2	9+ 5	1	3
3 3	1	3÷ 6	2	15+ 4	5
20× 5	4	6+ 1	3	2	6
1 1	1− 5	4	36× 6	3	2

2÷ 6	3	5× 1	5	4	12+ 2
40× 5	4	3 3	7+ 1	2	6
2	7+ 1	14+ 6	3	5	4
1 1	2	4	3− 6	3	5× 5
13+ 3	11+ 5	2	4	10+ 6	1
4	6	10× 5	2	1	3

6 6	2	9+ 3	5	1 1	7+ 4
12+ 4	6	10× 2	1	14+ 5	3
20× 1	4	5	2 2	3	6
5	2÷ 3	6	2÷ 4	2	1 1
6+ 3	10+ 5	1	15+ 6	40× 4	2
2	1	4	3	6	5

6 6	2× 2	16× 4	1	90× 3	5
2÷ 2	1	3 3	4	5	6
4	50× 5	2	30× 6	1	6× 3
15× 1	12× 4	5	3 3	24× 6	2
5	3	60× 6	2	4	4÷ 1
3	6÷ 6	1	5	2 2	4

20× 4	5	24× 1	12× 3	3÷ 2	6
30× 5	1 1	6	4	6× 3	2
3	2	4	1	30× 6	5
3÷ 6	90× 3	5	40× 2	4× 1	4
2	6	3 3	5	4	1
4÷ 1	4	60× 2	6	5	3 3

18× 3	24× 4	6	5 5	2× 1	2
2	3	96× 4	6	10× 5	1
30× 5	6	1	4	2	3 3
24× 6	15× 5	2 2	12× 1	3	4
4	1	3	24× 2	180× 6	5
1	10× 2	5	3	4	6

19 Medium 6×6 Solutions

2 — 2	24× — 1	6	3 — 3	20× — 4	5
12+ — 1	3	4	13+ — 6	5	2
6	2	15× — 3	10+ — 5	1	4
19+ — 4	6	5	7+ — 2	14+ — 3	1 — 1
5	4	2× — 2	1	6	3
15× — 3	5	1	4	2	6 — 6

5× — 5	1	2	3 — 3	5− — 6	12+ — 4
9+ — 6	11+ — 4	3	2	1	5
1	5 — 5	16+ — 6	4	2	3
2	6	4	5	3	1 — 1
4 — 4	3	15+ — 1	6	10× — 5	2
14+ — 3	2	16+ — 5	1	4	6

9+ — 1	2	2÷ — 3	6	5 — 5	4
11+ — 5	4	2	2× — 1	16+ — 6	3
6	15+ — 5	4	2	3	5− — 1
15+ — 3	1	5	9+ — 4	2	6
4	3 — 3	5− — 6	15× — 5	1	2
2	6	1	3	1− — 4	5

13+ — 4	11+ — 6	5	1 — 1	6+ — 3	2
3	11+ — 2	4	5	4− — 6	1
6	10+ — 5	1	3	2	12+ — 4
30× — 5	1	12+ — 2	6	4	3
2	3	18× — 6	16+ — 4	1	5
4÷ — 1	4	3	2 — 2	5	6

3 — 3	18+ — 5	1− — 4	11+ — 6	1	2
2× — 1	3	5	4	11+ — 2	6
2	4	6	30× — 5	17+ — 3	1
1− — 5	2× — 1	2	3	6	4 — 4
4	12+ — 6	1	2	5	3
6 — 6	2	3	1 — 1	9+ — 4	5

20+ — 6	6× — 2	3	4÷ — 1	4 — 4	5
3	5	6	4	8+ — 1	2
4÷ — 4	1	2 — 2	6× — 3	5 — 5	6
14+ — 5	3× — 3	1	2	19+ — 6	4
1	6	11+ — 4	5	2	3
2	1− — 4	5	6 — 6	3× — 3	1

20 Difficult 6×6 Solutions

(1) 1	(11+) 3	(11+) 6	(12+) 5	4	(3+) 2
(11+) 4	2	5	3	(10+) 6	1
5	6	(6+) 2	4	1	3
2	(10+) 5	1	(6) 6	(1−) 3	4
(3−) 6	4	(8+) 3	1	(13+) 2	5
3	(1) 1	4	(7+) 2	5	6

(11+) 2	3	6	(15+) 5	4	1
(4−) 1	5	3	6	(7+) 2	4
(9+) 4	(10+) 6	1	(10+) 3	5	2
5	(2) 2	(11+) 4	(3−) 1	(9+) 6	3
(11+) 6	1	2	4	3	(11+) 5
(3) 3	4	5	(6+) 2	1	6

(9+) 2	(14+) 6	5	3	(5+) 1	4
1	(12+) 2	4	6	(12+) 3	5
6	(5) 5	1	(6+) 2	4	3
(5) 5	(5+) 1	3	4	(11+) 2	6
(10+) 4	3	(3−) 2	5	(12+) 6	1
3	(11+) 4	6	1	5	(2) 2

(3+) 1	2	(15+) 6	5	4	(3) 3
(11+) 5	1	(9+) 4	2	(15+) 3	6
(4) 4	5	(2) 2	3	6	(1) 1
(14+) 3	6	5	1	(7+) 2	(11+) 4
(6) 6	3	(6+) 1	4	5	2
(9+) 2	4	(10+) 3	6	1	5

(6) 6	2	(9+) 3	5	(1) 1	(7+) 4
(12+) 4	6	(3−) 2	1	(14+) 5	3
(10+) 1	4	5	(2) 2	3	6
5	(9+) 3	6	(2−) 4	2	(1) 1
(6+) 3	(10+) 5	1	(15+) 6	(11+) 4	2
2	1	4	3	6	5

(5) 5	(10+) 6	(8+) 4	(8+) 1	2	(5+) 3
(11+) 6	4	1	3	5	2
4	(3) 3	(13+) 2	(11+) 6	(10+) 1	5
1	(3+) 2	6	5	(3) 3	4
(1−) 3	1	5	(12+) 2	4	(5−) 6
2	(12+) 5	3	4	6	1

21 Easy 9×9 Solution

$48\times$ 8	$6\div$ 6	$80\times$ 2	5	$42\times$ 7	$189\times$ 3	9	$36\times$ 1	4
6	1	4	8	3	2	7	5	9
$144\times$ 4	$216\times$ 9	6	$15\times$ 3	5	1 1	$70\times$ 2	7	$16\times$ 8
9	4	1 1	$336\times$ 7	8	$45\times$ 5	$144\times$ 3	6	2
$147\times$ 3	7	$45\times$ 5	4 4	6	9	8	2 2	1
7	8 8	9	1	$3\div$ 2	6	$5\times$ 5	$36\times$ 4	3
$10\times$ 2	5	$144\times$ 8	6	$36\times$ 9	4	1	3	$35\times$ 7
$10\times$ 1	2	3	$18\times$ 9	$224\times$ 4	7	$24\times$ 6	$72\times$ 8	5
5	$21\times$ 3	7	2	1 1	8	4	9	6 6