Induction Problems

Tom Davis
tomrdavis@earthlink.net
http://www.geometer.org/mathcircles
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All of the following problems should be proved by mathematical induction. The problems are not necessarily arranged in order of increasing difficulty.

1 Problems

1. Show that $3^n \geq 2^n$.
2. Prove that for any $n \geq 2$:
$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} < 1.$$
3. Prove that for any $n > 0$:
$$1^2 + 4^2 + 7^2 + 10^2 + \cdots + (3n-2)^2 = n(6n^2 - 3n - 1)/2.$$
4. Show that if n lines are drawn on the plane so that none of them are parallel, and so that no three lines intersect at a point, then the plane is divided by those lines into $(n^2 + n + 2)/2$ regions.
5. Show that if the same lines as in problem 4 are drawn on a plane that it is possible to color the regions formed with only two colors so that no two adjacent regions share the same color.
6. Assume that any simple (but not necessarily convex) n-gon (where $n > 3$) has at least one diagonal that lies completely within the n-gon. Show that any n-gon can be subdivided into exactly $n - 2$ triangles so that every triangle vertex is one of the original vertices of the n-gon.
7. Prove by induction the formula for the sum of a geometric series:
$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{r^n - 1}{r - 1}.$$
8. Show that:
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$
9. Suppose that you begin with a chocolate bar made up of n squares by k squares. At each step, you choose a piece of chocolate that has more than two squares and snap it in two along any line, vertical or horizontal. Eventually, it will be reduced to single squares. Show by induction that the number of snaps required to reduce it to single squares is $nk - 1$.

10. Show that for any $n \geq 2$:

$$\sqrt{2\sqrt{3\sqrt{4 \cdots (n-1)\sqrt{n}}}} < 3.$$

11. Show that 3^{n+1} divides evenly into $2^{3^n} + 1$ for all $n \geq 0$.

12. Define the Fibonacci numbers $F(n)$ as follows:

$$\begin{aligned} F(0) &= 0 \\ F(1) &= 1 \\ F(n) &= F(n-1) + F(n-2), \text{if } n > 1. \end{aligned}$$

Show that if $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$ then:

$$F(n) = \frac{a^n - b^n}{a - b}.$$

13. Show that if $F(n)$ is defined as in problem 12 and $n > 0$, then $F(n)$ and $F(n+1)$ are relatively prime.

14. Show that if $\sin x \neq 0$ and n is a natural number, then:

$$\cos x \cdot \cos 2x \cdot \cos 4x \cdots \cos 2^{n-1}x = \frac{\sin 2^n x}{2^n \sin x}.$$

15. Suppose there are n identical cars on a circular track and among them there is enough gasoline for one car to make a complete loop around the track. Show that there is one car that can make it around the track by collecting all of the gasoline from each car that it passes as it moves.

16. Show using induction that:

$$\sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} = 2^n.$$

2 Solutions

1. Show that $3^n \geq 2^n$.

   Proof: If $n = 0$, $1 = 3^0 \geq 2^0 = 1$, so it's true in this case.

   Assume $3^k \geq 2^k$. Multiply both sides by 3:

$$\begin{aligned} 3 \cdot 3^k &\geq 3 \cdot 2^k \\ 3^{k+1} &\geq (2+1) \cdot 2^k \\ 3^{k+1} &\geq 2 \cdot 2^k + 1 \cdot 2^k \\ 3^{k+1} &\geq 2^{k+1} + 2^k. \end{aligned}$$

   But since $2^k \geq 0$ we know that:

$$2^{k+1} + 2^k \geq 2^{k+1}$$

   so

$$3^{k+1} \geq 2^{k+1} + 2^k \geq 2^{k+1}.$$

2. Prove that for any $n \geq 2$:

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} < 1.$$

   Proof: First we will show that for $n \geq 2$:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1) \cdot n} = \frac{n-1}{n}.$$

   This is fairly straight-forward, but it does hide a small technical detail. Mathematical induction proves theorems for *all* natural numbers, starting at zero. But the statement of this problem includes the constraint that n be 2 or larger. Thus it is obviously true for $n = 0$ and for $n = 1$, but to make the induction work, we still have to show that it works specifically for $n = 2$ before we can continue with the proof that truth for $n = k$ implies truth for $n = k + 1$. Do you see why?

   If $n = 0$ and $n = 1$ it is true since there is no condition to be satisfied. If $n = 2$: $1/2 = 1/2$.

   If, for some $k \geq 2$ we know that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(k-1) \cdot k} = \frac{k-1}{k}$$

   then add $1/(k(k+1))$ to both sides:

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k-1) \cdot k} + \frac{1}{k \cdot (k+1)} &= \frac{k-1}{k} + \frac{1}{k \cdot (k+1)} \\ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k-1) \cdot k} + \frac{1}{k \cdot (k+1)} &= \frac{(k-1)(k+1)}{k(k-1)} + \frac{1}{k(k+1)} \\ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k-1) \cdot k} + \frac{1}{k \cdot (k+1)} &= \frac{k^2 - 1 + 1}{k \cdot (k+1)} \\ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k-1) \cdot k} + \frac{1}{k \cdot (k+1)} &= \frac{k}{k+1}. \end{aligned}$$

3. Prove that for any $n > 0$:

$$1^2 + 4^2 + 7^2 + 10^2 + \cdots + (3n-2)^2 = n(6n^2 - 3n - 1)/2.$$

Proof: For $n = 1$ we have: $1^2 = 1(6 - 3 - 1)/2$, so that works.

If we assume that it's true for $n = k$, we have:

$$1^2 + 4^2 + \cdots + (3k-2)^2 = k(6k^2 - 3k - 1)/2.$$

Add $(3(k+1)-2)^2 = (3k+1)^2$ to both sides and simplify:

$$\begin{aligned}
1^2 + \cdots + (3(k+1)-1)^2 &= k(6k^2 - 3k - 1)/2 + (3k+1)^2 \\
1^2 + \cdots + (3(k+1)-1)^2 &= (6k^3 - 3k^2 - k + 18k^2 + 12k + 2)/2 \\
1^2 + \cdots + (3(k+1)-1)^2 &= (6k^3 + 15k^2 + 11k + 2)/2 \\
1^2 + \cdots + (3(k+1)-1)^2 &= (6(k^3 + 3k^2 + 3k + 1) - 3(k^2 + 2k + 1) - (k+1))/2 \\
1^2 + \cdots + (3(k+1)-1)^2 &= (6(k+1)^3 - 3(k+1)^2 - (k+1))/2 \\
1^2 + \cdots + (3(k+1)-1)^2 &= (k+1)(6(k+1)^2 - 3(k+1) - 1)/2.
\end{aligned}$$

The final line above is equivalent to the original statement of the problem with $k+1$ substituted for n.

4. Show that if n lines are drawn on the plane so that none of them are parallel, and so that no three lines intersect at a point, then the plane is divided by those lines into $(n^2 + n + 2)/2$ regions.

Proof: First, if $n = 0$, we note that if zero lines are drawn on the plane, there is a single region: the whole plane, so the statement is true if $n = 0$ since $(0^2 + 0 + 2)/2 = 1$.

Next, assume that no matter how you draw k lines on the plane, consistent with the conditions of the problem, that there are exactly $(k^2 + k + 2)/2$ regions formed. Consider a similar configuration of $k+1$ lines. If we choose one of them and eliminate it, there will, according to the induction hypothesis, be $(k^2 + k + 2)/2$ regions. When we look at the line we temporarily eliminated, since it is not parallel to any of the lines, it must intersect all of them: that makes k intersections. None of these intersections are at the same point on the new line, or otherwise there would be three lines intersecting at a point, which is not allowed according to the conditions of the problem.

The points of intersection thus divide the new line into $k+1$ segments, each of which lies in a different one of the $(k^2 + k + 2)/2$ regions formed by the original k lines. That means that each of these $k+1$ segments divides its region into two, so the addition of the $(k+1)^{\text{st}}$ line adds $k+1$ regions. Thus there are now: $(k^2 + k + 2)/2 + (k+1)$ regions in the new configuration. A tiny bit of algebra shows that this is equivalent to $((k+1)^2 + (k+1) + 2)/2$ regions, which is what we needed to show.

Based on this idea, can you see how to find a formula for the number of regions into which space will be divided by n planes where the planes are in "general position", meaning that none are parallel, no three of them pass through the same line, and no four of them through the same point? Hint: what does the intersection of the $(k+1)^{\text{st}}$ plane with k planes look like?

5. Show that if the same lines as in problem 4 are drawn on a plane that it is possible to color the regions formed with only two colors so that no two adjacent regions share the same color.

   Proof: With zero lines, you can obviously do it; in fact, one color would be sufficient. If you can successfully 2-color the plane with k lines, when you add the $(k+1)^{\text{st}}$ line, swap the colors of all the regions on one side of the line. This will provide a 2-coloring of the configuration with $k+1$ lines. (In fact, for this problem, there is no real need to have the lines in general position: some can be parallel, and multiple lines can pass through a point, and the proof will continue to work.)

6. Assume that any simple (but not necessarily convex) n-gon (where $n > 3$) has at least one diagonal that lies completely within the n-gon. Show that any n-gon can be subdivided into exactly $n-2$ triangles so that every triangle vertex is one of the original vertices of the n-gon.

   Proof: This one starts at $n = 3$; for $n < 3$ it is true by default, since there is no such thing as a 2-gon or 1-gon. For $n = 3$, a 3-gon is a triangle, and it is obviously possible to make a triangle into a triangle by drawing exactly 0 diagonals.

   Next, assume $k \geq 3$ and that every m-gon, where $m \leq k$, can be divided into $m-2$ triangles with internal diagonals. (Note that this uses general induction: we need more than just the fact that k-gons can be subdivided.)

   Consider any $(k+1)$-gon. Since $k+1$ is at least 4, by our hypothesis (which can be proved, by the way, but the proof doesn't depend on induction, so we omit it here) we can find an internal diagonal that will divide the original $(k+1)$-gon into two polygons. Ignoring the diagonal, one with contain m_1 of the original edges of the polygon and the other will contain $(k+1-m_1)$ edges. The new diagonal will add an edge to both, so we wind up with two polygons, one containing m_1+1 edges and the other, $k+2-m_1$ edges. The first (by the induction hypothesis) can be divided into m_1+1-2 triangles and similarly, the other can be subdivided into $k+2-m_1-2$ triangles. Add those triangles together, and we have $k-1 = (k+1)-2$ triangles, so the proof is complete.

7. Prove by induction the formula for the sum of a geometric series:

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{r^n - 1}{r - 1}.$$

   Proof: For $n = 1$ we need to show that $a = a(r-1)/(r-1)$. Note that this even works if $r = 0$.

   Assume it's true for $n = k$:

$$a + ar + ar^2 + \cdots + ar^{k-1} = a\frac{r^k - 1}{r - 1},$$

   and add ar^k to both sides:

$$\begin{aligned}
a + ar + ar^2 + \cdots + ar^{k-1} + ar^k &= a\frac{r^k - 1}{r - 1} + ar^k \\
a + ar + ar^2 + \cdots + ar^{k-1} + ar^k &= a\Big(\frac{r^k - 1}{r - 1} + \frac{(r-1)r^k}{r - 1}\Big) \\
a + ar + ar^2 + \cdots + ar^{k-1} + ar^k &= a\Big(\frac{r^k - 1 + (r-1)r^k}{r - 1}\Big)
\end{aligned}$$

$$
\begin{aligned}
a + ar + ar^2 + \cdots + ar^{k-1} + ar^k &= a\Big(\frac{r^{k+1} + r^k - r^k - 1}{r-1}\Big) \\
a + ar + ar^2 + \cdots + ar^{k-1} + ar^k &= a\frac{r^{k+1} - 1}{r-1},
\end{aligned}
$$

which is what we needed to show.

8. Show that:
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

Proof: If $n = 1$: $1^3 = (1)^2$ so that part is ok.

Assume it's true for $n = k$:
$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2,$$
and add $(k+1)^3$ to both sides:
$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = (1 + 2 + 3 + \cdots + k)^2 + (k+1)^3.$$

Replace the $1 + 2 + 3 + \cdots + k$ on the right with $k(k+1)/2$, since this is just the formula for the sum of an arithmetic series (which could itself be proved by induction if you did not already know the formula). This yields:

$$
\begin{aligned}
1^3 + 2^3 + \cdots + (k+1)^3 &= (k(k+1)/2)^2 + (k+1)^3 \\
1^3 + 2^3 + \cdots + (k+1)^3 &= \Big(\frac{k^4 + 2k^3 + k^2}{4} + \frac{4(k+1)^3}{4}\Big) \\
1^3 + 2^3 + \cdots + (k+1)^3 &= \frac{k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4}{4} \\
1^3 + 2^3 + \cdots + (k+1)^3 &= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \\
1^3 + 2^3 + \cdots + (k+1)^3 &= \frac{(k+1)^2(k+2)^2}{4} \\
1^3 + 2^3 + \cdots + (k+1)^3 &= ((k+1)(k+2)/2)^2 = (1 + 2 + \cdots + (k+1))^2,
\end{aligned}
$$

and this is exactly what we need to prove the result. The factorization of the forth-degree polynomial might seem difficult, but it's not since we know *exactly* what we're looking for!

9. Suppose that you begin with a chocolate bar made up of n squares by k squares. At each step, you choose a piece of chocolate that has more than two squares and snap it in two along any line, vertical or horizontal. Eventually, it will be reduced to single squares. Show by induction that the number of snaps required to reduce it to single squares is $nk - 1$.

Proof: The "nk" in this problem is misleading. Although it might be possible to induct on both n and k in some way to solve the problem, it's much easier to restate the problem as concerning a rectangular chocolate bar with n pieces. With that restatement, we'll solve the problem.

If $n = 1$, then zero breaks are required, and $0 = 1 - 1$, so the result is correct.

Assume for some k that the statement is true for every chocolate bar with k or fewer squares. Consider a chocolate bar with $k+1$ total pieces. After the first snap, there will be two smaller bars: one will have m squares and the other, $k+1-m$. Both of these bars must have k or fewer squares, so the first will require $m-1$ snaps and the other, $k+1-m-1$ snaps to reduce to individual squares. The total snaps include both of these numbers plus the one snap required to break the original bar of $k+1$ squares, for a total of $(m-1)+(k+1-m-1)+1 = k$ snaps; one less than $k+1$, which is what we were required to show.

Note that the breaks do not have to be straight lines; any kind of crooked break will work, as long is it divides a piece into two pieces. Also note that an easier proof than the inductive one above might be this: Each snap produces one more piece, and we start with 1 and end with nk. Therefore there must be $nk-1$ snaps.

10. Show that for any $n \geq 2$:
$$\sqrt{2\sqrt{3\sqrt{4\cdots(n-1)\sqrt{n}}}} < 3.$$

Proof: This problem is also tricky, and you may be tempted to induct on n. Actually, it is easier to solve the following problem of which the problem above is a special case. Show that for any $m < n$, that:
$$\sqrt{m\sqrt{(m+1)\sqrt{(m+2)\cdots(n-1)\sqrt{n}}}} < m+1.$$

If we set $m = 2$ we have our result.

We can prove this by induction on "how far m is before n". If $n = m$, the starting case that would correspond to what we normally look at as the $n = 0$ case, we need to show that: $\sqrt{m} < m+1$ and this is true, even if $m = 1$.

Now to get to the case with m, we assume it's true for $m+1$ (remember, we're going backwards: it is possible to restate this in terms of an increasing value by looking at $k = n-m$ and you might want to do that to be sure you understand what's going on). Assuming it's true for $m+1$, we have:
$$\sqrt{(m+1)\sqrt{(m+2)\cdots(n-1)\sqrt{n}}} < m+2.$$

Multiply both sides by m and take the square root and we obtain:
$$\sqrt{m\sqrt{(m+1)\sqrt{(m+2)\cdots(n-1)\sqrt{n}}}} < \sqrt{m(m+2)},$$

so to complete the proof, we just need to show that:
$$\sqrt{m(m+2)} < m+1.$$

If we square both sides, we obtain the obviously true result:
$$m^2 + 2m < m^2 + 2m + 1,$$

and we are done.

11. Show that 3^{n+1} divides evenly into $2^{3^n} + 1$ for all $n \geq 0$.

Proof: If $n = 0$, we need to show that $3|3$, which it obviously does. (In case you haven't seen it, the expression "$k|m$" means that k divides evenly into m.)

Assume that for some k, $3^{k+1}|(2^{3^k}+1)$. Given that this is true, we need to show that $3^{k+2}|2^{3^{k+1}}+1$. But the term on the right is just:

$$(2^{3^k})^3 + 1,$$

which is the sum of two cubes and can be factored:

$$(2^{3^k})^3 + 1 = (2^{3^k} + 1)(2^{2\cdot 3^k} - 2^{3^k} + 1).$$

By the induction hypothesis, the term on the left (the $(2^{3^k} + 1)$)is divisible already by 3^{k+1}. To complete the proof, we only need to show that the term on the right is divisible by 3. In other words, we need to show that:

$$3|2^{2\cdot 3^k} - 2^{3^k} + 1.$$

We can essentially do this by induction, too. We will show that $2^{3^k} = 2 (\text{mod } 3)$ so its square, $2^{2\cdot 3^k} = 1(\text{mod } 3)$ and that will make the sum above equal to $1 - 2 + 1 = 0(\text{mod } 3)$.

For $k = 0$, $2^{3^0} = 2 = 2(\text{mod } 3)$. If it's true for some arbitrary k, then the expression with $k + 1$ will simply be the cube of the value, so $2^{3^{k+1}} = (2^{3^k})^3 = 2^3(\text{mod } 3) = 8(\text{mod } 3) = 2(\text{mod } 3)$, so we are done.

12. Define the Fibonacci numbers $F(n)$ as follows:

$$\begin{aligned} F(0) &= 0 \\ F(1) &= 1 \\ F(n) &= F(n-1) + F(n-2), \text{if } n > 1. \end{aligned}$$

Show that if $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$ then:

$$F(n) = \frac{a^n - b^n}{a - b}.$$

Proof: A little algebra shows that this is obviously true for $n = 0$ and $n = 1$. Assume that it is true for $n = k > 1$. That means that the following two equations hold:

$$\begin{aligned} F(k-1) &= \frac{a^{k-1} - b^{k-1}}{a - b} \\ F(k) &= \frac{a^k - b^k}{a - b}. \end{aligned}$$

Add them together:

$$\begin{aligned} F(k+1) = F(k-1) + F(k) &= \frac{a^{k-1} + a^k - b^{k-1} + b^k}{a - b} \\ F(k+1) &= \frac{a^{k-1}(1 + a) - b^{k-1}(1 + b)}{a - b}, \end{aligned}$$

but a and b are the two solutions to the equation $x^2 = x + 1$ (this is easy to check), so $1 + a = a^2$ and $1 + b = b^2$. Substitute these into the final equation above to obtain:

$$F(k+1) = \frac{a^{k+1} - b^{k+1}}{a - b},$$

which is what we needed to show.

13. Show that if $F(n)$ is defined as in problem 12 and $n > 0$, then $F(n)$ and $F(n+1)$ are relatively prime.

    Proof: This is easy to check for the first couple of values of n, so assume that it is true for $n = k > 2$, so $F(k)$ and $F(k+1)$ are relatively prime. We would like to show that $F(k+2)$ and $F(k+1)$ are also relatively prime. To do that, we will just use the Euclidean algorithm to find the greatest common divisor of $F(k+2)$ and $F(k+1)$. If that is 1, we are done.

    Consider how the Euclidean algorithm looks for this pair:

$$\begin{aligned} F(k+2) &= d_1 \cdot F(k+1) + r_1 \\ F(k+1) &= d_2 \cdot r_1 + r_2 \\ r_1 &= d_3 \cdot r_2 + r_3 \\ \ldots &= \ldots \end{aligned}$$

    where r_i are the remainders and the d_i are the divisors. The way the algorithm works is that each pair of remainders has the same common denominator as the pair above. But by the definition of F, the $d_1 = 1$ and $r_1 = F(k)$ so $F(k+2)$ and $F(k+1)$ must have the same greatest common divisor as $F(k+1)$ and $F(k)$ which the induction hypothesis tells us is equal to 1, so we are done.

14. Show that if $\sin x \neq 0$ and n is a natural number, then:

$$\cos x \cdot \cos 2x \cdot \cos 4x \cdots \cos 2^{n-1}x = \frac{\sin 2^n x}{2^n \sin x}.$$

    Proof: If $n = 1$ we need to show that

$$\cos x = \frac{\sin 2x}{2sinx},$$

    but that is just a trivial consequence of $\sin 2x = 2 \sin x \cos x$.

    Assume it is true for $n = k$:

$$\cos x \cdot \cos 2x \cdot \cos 4x \cdots \cos 2^{k-1}x = \frac{\sin 2^k x}{2^k \sin x}$$

    and multiply both sides by $\cos 2^k x$:

$$\cos x \cdot \cos 2x \cdot \cos 4x \cdots \cos 2^k x = \frac{\sin 2^k x \cos 2^k x}{2^k \sin x}.$$

    The numerator of the fraction on the right is just $\sin 2^{k+1}x/2$, so we are done.

15. Suppose there are n identical cars on a circular track and among them there is enough gasoline for one car to make a complete loop around the track. Show that there is one car that can make it around the track by collecting all of the gasoline from each car that it passes as it moves.

    Proof: If there is a single car ($n = 1$), and that car has enough gas to make it around the loop, then it can make it around the loop, and we are done.

    Assume the statement of the problem is true for $n = k$ cars, and consider a situation with $k + 1$ cars on the loop. If we can find a particular car C_1 that has enough gas to make it to the next car C_2 then the situation is equivalent to transferring all of C_2's gas to C_1 and eliminating C_2 from the track. With one car eliminated, we know that there is a car that can complete the circuit from the induction hypothesis.

    Suppose that no car has enough gas to make it to the next one. Then if each car drove forward as far as it could, there would be a gap in front of each one, so the total amount of gas would not be sufficient for one car to make it all the way around the track. Thus there must be a car that can make it to the next, and the proof is complete.

    When this problem was submitted in the first math circle, one of the students came up with a different solution that required no induction. Here it is:

    Imagine the cars are on the track as before, but at first, instead of thinking of them as cars, just think of them as little fuel tanks. Add another phantom car to the track at any point with exactly enough fuel in its tank to make one complete loop. It can surely make it around, but as it goes around, imagine that it collects gas from each of the original cars. When it finishes, it will have exactly as much fuel as when it started, since it collected exactly the amount it burned during the loop.

    The amount of gas this phantom car has varies throughout the trip, having relative minima as it reaches each car/fuel-tank, where the fuel supply jumps up again. Pick the place where the phantom car has minimum gas. Clearly, if the phantom car had started there with an empty tank (at which point it would immediately get the gas from the minimal car), it could complete the loop since its tank would never get lower than it was at that point; namely, empty. That means that the car where this occurs could complete the route as required by the original problem.

    Figure 1 may help visualize this. The graph at the bottom represents the amount of gas in the tank of a phantom car that picks up no fuel on the way. It begins at A, and ends at B (which is just the same as A after a complete loop). The length AX represents enough fuel to make one loop, so after completing the loop, the tank is empty.

    In the upper graph in the same figure, the phantom car begins with the same amount of fuel, but now there are four cars at C_1, C_2, C_3 and C_4. The lengths C_1D_1, et cetera represent the amount of gas in each of the cars. The four lengths, C_1D_1, C_2D_2 and so on, added together, are exactly equal in length to AX. This time, as soon as the phantom car's position reaches each of the real cars C_i, its fuel increases by the amount by C_iC_i. Clearly, it'll have a full tank at the end, so will be again at height X.

    For this particular configuration, P_4 is the point at which the phantom car's tank is as low as it will ever get. If, instead of having C_4P4 fuel at that point (at which point it would pick up the fuel from the fourth car), its tank were empty, it could still make the loop, since we know its fuel would never get below that. Thus car 4 could also make it, since it is in exactly the same condition as the phantom car, having just run out of fuel, and then taken on board car 4's gasoline. Since the phantom car can make it, so will the fourth car.

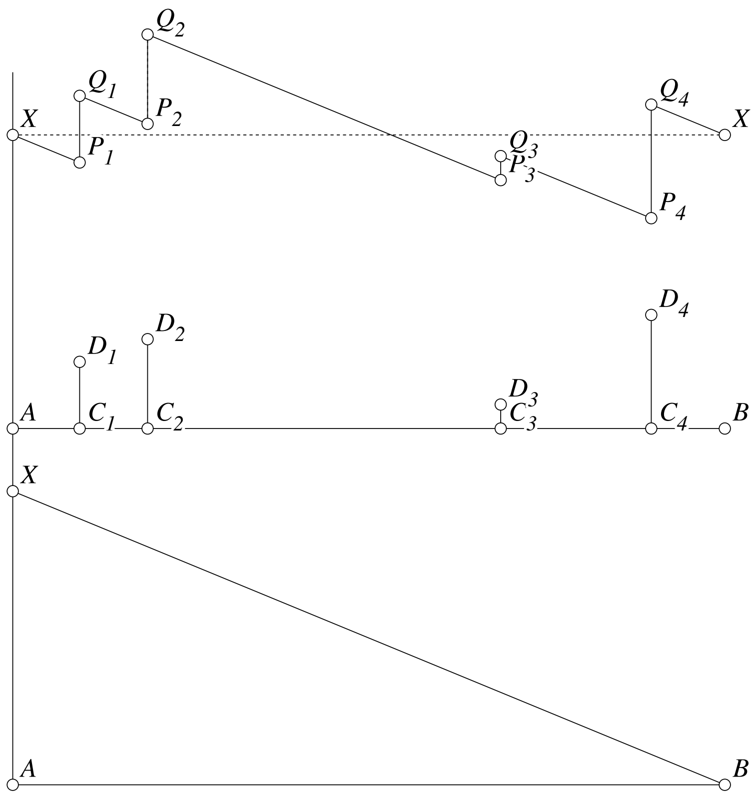

Figure 1: Phantom Car Fuel Tank

16. Show using induction that:

$$\sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} = 2^n.$$

Proof: Let $S(n) = \sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} = 2^n$. If $n = 0$ then we need to show that:

$$S(0) = \binom{0}{0} \cdot \frac{1}{1} = 2^0 = 1,$$

which is obviously true.

The proof will be based on the fact that for any $m \geq p > 0$:

$$\binom{m+1}{p} = \binom{m}{p-1} + \binom{m}{p}. \qquad (1)$$

Assume that it is true for $n = j$:

$$S(j) = \sum_{k=0}^{j} \binom{j+k}{k} \frac{1}{2^k} = 2^j,$$

and using equation 1 in every term below we obtain:

$$S(j+1) = \sum_{k=0}^{j+1} \binom{j+1+k}{k} \frac{1}{2^k} = \sum_{k=0}^{j+1} \binom{j+k}{k-1} \frac{1}{2^k} + \sum_{k=0}^{j+1} \binom{j+k}{k} \frac{1}{2^k}. \tag{2}$$

When $k = 0$ in the first term on the right in equation 2 above, ($\sum_{k=0}^{j+1} \binom{j+k}{k-1} \frac{1}{2^k}$) the first term of the summation is zero, so we can write that sum as if it began at $k = 1$. In the second term on the right, strip off the term corresponding to $k = j + 1$ and write it separately:

$$S(j+1) = \sum_{k=1}^{j+1} \binom{j+k}{k-1} \frac{1}{2^k} + \sum_{k=0}^{j} \binom{j+k}{k} \frac{1}{2^k} + \binom{2j+1}{j+1} \frac{1}{2^{j+1}}. \tag{3}$$

In the first term on the right of equation 3, we can change the summation variable with the substitution $k = l + 1$ so it will then go from $l = 0$ to j. We also moved the rightmost term to the middle:

$$S(j+1) = \sum_{l=0}^{j} \binom{j+1+l}{l} \frac{1}{2^{l+1}} + \binom{2j+1}{j+1} \frac{1}{2^{j+1}} + \sum_{k=0}^{j} \binom{j+k}{k} \frac{1}{2^k}. \tag{4}$$

In equation 4 the rightmost sum is $S(j)$. The second term can be rewritten:

$$\begin{aligned}
\binom{2j+1}{j+1} \frac{1}{2^{j+1}} &= \frac{(2j+1)\cdots(j+1)}{(j+1)!} \frac{1}{2^{j+1}} \\
&= \frac{(2j+2)\cdots(j+1)}{(2j+2)(j+1)!} \frac{1}{2^{j+1}} \\
&= \frac{(2j+2)\cdots(j+1)}{(j+1)(j+1)!} \frac{1}{2^{j+2}} \\
&= \frac{(2j+2)\cdots(j+2)}{(j+1)!} \frac{1}{2^{j+2}} \\
&= \binom{2j+2}{j+2} \frac{1}{2^{j+2}}.
\end{aligned}$$

Making those two substitutions, we obtain:

$$S(j+1) = \sum_{l=0}^{j} \binom{j+1+l}{l} \frac{1}{2^{l+1}} + \binom{2j+2}{j+2} \frac{1}{2^{j+2}} + S(j). \tag{5}$$

The combinatorial coefficient can be combined with the summation in equation 5 to obtain:

$$S(j+1) = \sum_{l=0}^{j+1} \binom{j+1+l}{l} \frac{1}{2^{l+1}} + S(j).$$

But the summation above now represents $S(j+1)/2$, so:

$$S(j+1) = S(j+1)/2 + S(j),$$

which implies that $S(j+1) = 2S(j)$, and we are done.