Zome Patterns

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1 Introduction

The Zome system is a construction set based on a set of plastic struts and balls that can be attached together to form an amazing set of mathematically or artistically interesting structures.

For information on Zome and for an on-line way to order kits or parts, see:

`http://www.zometool.com`

The main Zome strut colors are red, yellow and blue and most of what we'll cover here will use those as examples. There are green struts that are necessary for building structures with regular tetrahedrons and octahedrons, and almost everything we say about the red, yellow and blue struts will apply to the green ones. The green ones are a little harder to work with (both physically and mathematically) because they have a pentagon-shaped head, but can fit into any pentagonal hole in five different orientations. With the regular red, yellow and blue struts there is only one way to insert a strut into a Zome ball hole.

For these exercises, we recommend that students who have not worked with the Zome system before restrict themselves to the blue, yellow and red struts. Students with some prior experience, or who seem to be very fast learners, can try building things (like a regular octahedron or a regular tetrahedron) with some green struts. Most people find that their first experience with green struts can be pretty discouraging, and even building their first regular tetrahedron can be a frustrating experience due to all the possible angles that can be formed and the fact that the green struts do not lock as solidly into the Zome balls.

2 Overview of the Exercises

Although the Zome system can be used to investigate many aspects of mathematics and geometry, the exercises in this document will be related to Euler's theorem.

Euler's theorem simply states that if you take a figure that is topologically equivalent to a sphere (this is explained in Section 4), if you count the number of vertices (V), edges (E) and faces (F), you will always find that $V - E + F = 2$. The value 2 for $V - E + F$ is known as the "Euler characteristic" of the sphere. By building structures that are sphere-like and counting V, E and F and making tables of the values, the relationship can be discovered. The discovery of the relationship, while interesting, is secondary to what you're really "tricking" the students into doing; namely, learning to count in a logical, organized way. Counting the number of edges, faces and vertices of a cube is easy (although a lot of students will probably miscount the edges), but for more complex figures, like a dodecahedron or icosahedron, counting can be tricky, and it's a good idea to have alternative ways to count to check the answers obtained.

In these exercises we will not formally prove Euler's theorem, but will provide a huge amount of evidence that it is true. If you are interested in a formal proof of the theorem, see:

`http://www.geometer.org/mathcircles/euler.pdf`

3 Zome Practice

Many students may never have seen the Zome system and for that reason, it is a good idea for them to practice building a few simple structures to get a feeling for how it works. The struts are tough, but not indestructible, so ask the students to take a little care to avoid breaking the plastic struts, especially when they are disassembling their projects. The best way to avoid damage is to be sure to pull the struts directly away from the Zome balls rather than applying any sideways torque.

Important note: Before you let the students leave the exercise, ask them to take apart any structures they have built. While it is easier to take Zome structures apart than to put them together, if you've got a dozen builders and only one dismantler (you), you will be very sorry at the end of the day!

Have everyone build the following three objects that require only blue struts all having the same lengths (the medium-length blue struts work very well): a cube, an icosahedron and a dodecahedron. Point out that in each of these three regular solids, every face is regular (so the faces of the cube are perfect squares, the faces of the icosahedron are equilateral triangles, and the faces of the dodecahedron are regular pentagons. Figure 1 illustrates those three figures. Students may have some problems with the dodecahedron because although it is relatively easy to construct a pentagon that forms the first face, each face has an "inside" and an "outside" and the "inside" of the pentagon has to be in the interior of the dodecahedron. In other words, if they have trouble, have them turn over their original pentagon and try again.

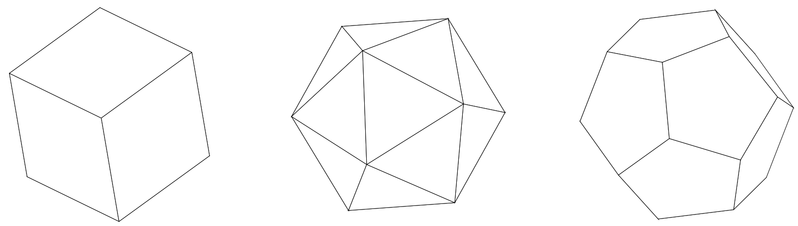


Figure 1: Cube, Icosahedron and Dodecahedron

4 Counting V, E and F

After the students have constructed those examples, have them count the number of vertices (Zome balls), edges (Zome struts) and faces (flat regions surrounded by balls and struts) and write their results in a table like the one below, where V, E and F refer to the counts of vertices, edges and faces. These counts can be done in various ways, and if the students can figure out how to count them in more than one way and obtain the same result, then they can be more confident that the count is correct. After the table has enough entries, lead the students to discover the fact that for every object they make that doesn't include "holes", that $V - E + F = 2$. (For now, we'll say that an object doesn't include holes if you could imagine inflating a balloon inside so that it completely filled the object without having to have the outside of the balloon touch itself. If you imagine putting a balloon inside a doughnut-shaped object and inflating it, you could fill the doughnut, but the balloon would have to come around and touch itself, so the doughnut has a hole.)

Object	Vertices (V)	Edges (E)	Faces (F)	
Cube	8	12	6	

Notice in the table above there is one extra unmarked column where we'll eventually write the formula $V - E + F$ and that column will be filled with the number 2. An example of what the students might come up with appears in Section 9. Finally, a worksheet for students appears on the final page of this document.

Some counting strategies that work include putting the object in a fixed orientation so the components of interest can be classified. For example, to count the edges in a cube, place it face-down on a table, and then there are 4 edges around the top and bottom faces, and 4 connecting the top and bottom, making a total of 12. To count the faces on a dodecahedron, when the object is placed with a face flat on the table, there are the top and bottom faces, 5 sharing an edge with the top and 5 sharing an edge with the bottom, for a total of 12. Have the students find similar strategies to count the features (edges, faces and vertices) on other objects.

Obviously, you can disassemble a structure and count the Zome balls to obtain V and the struts to obtain E, but you still need to count the faces carefully.

Another nice strategy that works for many objects (and these regular ones, in particular) is to imagine that the figure is made of cardboard and you cut it into its individual faces. For example, suppose you have figured out that the icosahedron has exactly 20 faces. If you cut it up, you will have 20 equilateral triangles. Each triangle has 3 edges, so the triangles together have $20 \times 3 = 60$ edges. But each cut of the original icosahedron converted one edge to two, so the number of edges in the original icosahedron contained $60/2 = 30$ edges. Similarly, the cut triangles each have 3 vertices, so after the cutting, there are $20 \times 3 = 60$ vertices. But if you look at the original icosahedron, 5 triangles meet at every vertex so when you cut it apart, each of the original vertices will appear 5 times, on 5 different triangles, so there are $60/5 = 12$ vertices in the icosahedron.

This "counting by cutting" strategy can be used on the cube and dodecahedron as well.

Here are the results the students should obtain. For the cube, $V = 8, E = 12, F = 6$. For the dodecahedron, $V = 20, E = 30, F = 12$, and for the icosahedron, $V = 12, E = 30, F = 20$.

5 More Examples

Have the students make more objects that do not contain holes (where "holes" are defined in the first paragraph of Section 4) and for each such object, have them count the number of edges, faces, and vertices and enter those numbers in their table. These new objects need not be regular, and make sure the students know that they are welcome to use struts of different lengths and colors.

If the students lack imagination, here are some things they can build. Tetrahedrons ($V = 4, E = 6, F = 4$) and octahedrons ($V = 6, E = 12, F = 8$). Regular tetrahedrons and octahedrons cannot be built without the green struts, but irregular ones, where the edges are different colors and lengths can be made.

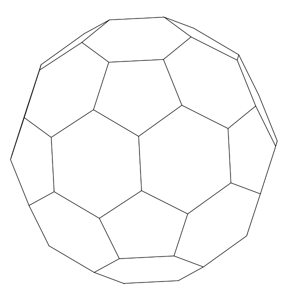


Figure 2: Soccer Ball

Other examples include pyramids or other "cone-like" figures with flat bases whose vertices are connected to the tip of the cone with struts. More complex examples include the “soccer ball” which is technically called a truncated icosahedron (see Figure 2, which can be built with blue and yellow struts that are closest in length to the blue ones), or even a copy of the Zome ball itself (this requires short and medium-length blue struts only). The truncated icosahedron counts are ($V = 60, E = 90, F = 32$) and the Zome ball’s counts are ($V = 60, E = 120, F = 62$). A modified version of the Zome ball where the rectangles are replaced by squares can be built entirely using only blue struts of a single length.

The features of both the truncated icosahedron and Zome ball can be counted using the “counting by cutting” strategy explained at the end of Section 4. The soccer ball has 12 pentagons and 20 hexagons, but exactly three edges meet at each vertex. If you cut it up, you’ll have 32 faces, $12 \times 5 + 20 \times 6 = 180$ edges and vertices. As before, all edges are double-counted, making the actual number of edges $180/2 = 90$, and all vertices are triple-counted, making the actual number $180/3 = 60$.

To count the faces of the Zome ball, see Section 8, but once you know there are 12 pentagonal faces, 20 triangular faces and 30 rectangular faces, you can see that in the cut-up version of a cardboard Zome ball, there will be $12 \times 5 + 20 \times 3 + 30 \times 4 = 240$ edges and vertices on the pieces after the cuts. The edges are double-counted yielding $240/2 = 120$ edges of the original Zome ball and since four struts meet at each vertex, they are quadruple-counted, yielding $240/4 = 60$ vertices.

Another strategy to use when constructing objects is to take one that is complete and modify it by adding features. For example, if you begin with a cube and extend one of the faces so that it looks like an Egyptian pyramid, you effectively eliminate the original face and replace it by the four faces of the pyramid, thus increasing the number of faces by $4 - 1 = 3$. Only one vertex is added and none are deleted, so the number of vertices increase by 1. Finally, there are 4 new edges, so the number of edges increase by 4. Since the original cube had $V = 8, E = 12, F = 6$, the new object will have $V = 8 + 1 = 9, E = 12 + 4 = 16, F = 6 + 3 = 9$. Notice that the net change this makes to $V - E + F$ is $1 - 4 + 3 = 0$. Thus if Euler’s theorem was true for the cube, it will be true for the new object. Not only that, but if it was true for *any* object with a rectangular face and that face was modified as we did the face on the cube, the new object will satisfy Euler’s theorem if the old one did.

Here’s another idea to think about. What if you make a Zome structure that satisfies Euler’s theorem, and then you build one that is exactly twice as big by replacing each strut in the original by a “strut-ball-strut” (effectively a strut that’s twice as long)? Well, the number of faces will remain the same, but if the original structure had n struts, the new one will have an additional n struts and n balls. Thus, both V and F will be increased by n, but this will leave the quantity $V - E + F$ unchanged.

6 Duality

If you look at the V, E and F counts of the cube and octahedron, you’ll notice that they are the same except that the values of V and F are exchanged. Exactly the same thing occurs with the dodecahedron-icosahedron pair. This does not occur by chance:

these pairs are called "duals" and a dual can be formed from many of the objects we can make.

To see more clearly what a dual object is, consider the following construction, and for concreteness, think about it in relation to the cube. Suppose the cube's vertices and edges are red. Place blue vertices in the center of each of the cube's faces. Next, connect the blue vertices that lie on adjacent faces with blue edges and consider the figure formed. If you look toward the center of the cube through each red edge, you will see exactly one new blue edge crossing it. Similarly, if you look toward the center of the cube through each red vertex you will see that a face surrounded by blue lines lies between the red vertex and the center of the cube. Thus, the newly-formed solid has exactly as many edges as the cube (12), it has the same number of faces as the cube had vertices (8) and the same number of vertices as the cube had faces (6). The newly-formed object is called an octahedron, and it is called the dual of the cube.

This explains why the numbers work out as they do, but what is also interesting is that the dual of the octahedron is the cube again. A similar thing occurs with the dodecahedron-icosahedron pair. The tetrahedron is its own dual. (It is called "self-dual", for that reason.)

Another interesting feature of dual structures that is obvious if you consider the construction above is that in the dodecahedron, for example, 3 faces meet at each vertex and 5 vertices surround each face. For its dual, the icosahedron, the opposite is true: 5 faces meet at each vertex and 3 vertices surround each face. For the cube-octahedron pair, the numbers are 3 and 4, and for the tetrahedron, which is self-dual, the numbers, of course must be the same: 3 and 3.

Have the students look for other dual structures. What would be the dual of an Egyptian pyramid, the soccer ball, or the Zome ball?

7 Objects with Holes

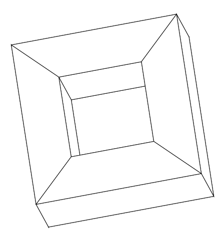


Figure 3: Doughnut

It is possible to construct objects with holes using Zome, and a more general version of Euler's theorem can be obtained. For example, if we construct a doughnut-like object (something like what is illustrated in Figure 3 which has $V = 16, E = 32, F = 16$), then we will find that $V - E + F = 0$. This will be true for any doughnut-like, or one-holed object. If an object has two holes, you will find that $V - E + F = -2$ and each additional hole subtracts 2 more from the Euler characteristic. In fact, the number of holes in a very complex object can be counted by finding the Euler characteristic for that object and from that deriving the number of holes. For us humans, it's pretty easy just to count the holes, but a computer trying to do the same for a mathematical description of the model would be hard-pressed to "see" the holes and would find it far easier just to calculate the Euler characteristic. If it found, for example, that $V - E + F = -22$, that would mean that the object had 12 holes.

It is not hard to do, but it takes a while to construct Zome objects with holes, so experimenting with such objects is probably only reasonable to do with students who already know about the simple version of Euler's theorem and who have some previous experience with the Zome system.

8 The Zome Ball

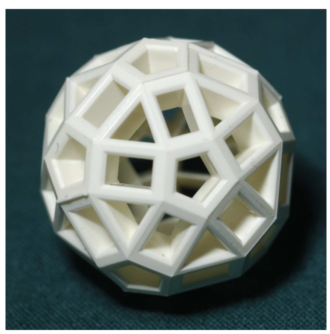


Figure 4: The Zome Ball

Look carefully at a Zome ball. (It is better to look at a physical ball, but an image of one appears in Figure 4. It is highly symmetric, and has holes that will accept struts of three shapes: rectangles with an aspect ratio of roughly $1 : 1.618$ (which happens to be the golden ratio), equilateral triangles and regular pentagons. Every pentagonal hole looks the same: it is surrounded by 5 rectangular holes and 5 triangular holes. The same can be said of every hole: the shapes and orientations of the neighboring holes are the same for every hole in the ball.

Another way to convince yourself that all the holes of a certain shape are basically identical is to place a Zome ball on a table balanced on a hole of a particular shape (say a rectangle). Now take another Zome ball and place it on *any* of its rectangular holes (or hole of the same shape as the first ball). If you rotate the second ball so that the rectangles on top have the same orientation, you will find that every hole matches in shape and orientation in the two balls.

There are 12 pentagon-shaped holes and if you imagine that the pentagons were all left in their planes but expanded until their edges touched the nearest pentagon edges, the resulting figure would be a dodecahedron (a regular 12-sided polyhedron as illustrated in Figure 1).

If you think about this pentagon expansion, every pair of adjacent pentagons would close over a rectangular hole, so there are the same number of rectangular holes as there are edges in a dodecahedron; namely, 30.

Finally, again visualizing the expansion of the pentagonal holes, each triangle on the Zome ball will be covered by a vertex of the final dodecahedron, so there are the same number of triangles as there are vertices of a dodecahedron; namely, 20.

A dual argument can be made: instead of expanding the pentagons until their edges merge, expand the triangles in the same way, and the resulting figure will be a regular icosahedron – a polyhedron with 20 identical triangluar sides. Each vertex of the resulting icosahedron (of which there are 12) corresponds to a pentagonal hole in the Zome ball and each edge of the icosahedron (of which there are 30) corresponds to one of the rectangular holes in the Zome ball.

Luckily, we obtain the same counts using both approaches: 12 pentagonal holes, 20 triangular holes and 30 rectangular holes for a total of $12 + 20 + 30 = 62$ holes.

9 Appendix

Here are some examples of what the students might include in their tables. The "n-Cylinder" is a sort of cylinder made by extruding an n-sided polygon so that it has two faces made of that polygon plus n rectangles connecting the corresponding sides. The n-Pyramid is a sort of cone with an n-sided polygonal base and all the base vertices are connected to a vertex that forms the tip of the cone. An Egyptian pyramid is made by applying this process to a square.

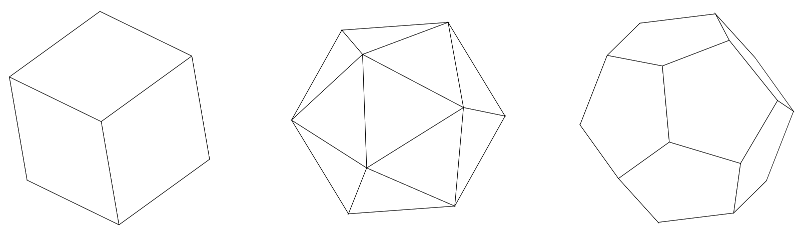
Object	Vertices (V)	Edges (E)	Faces (F)	$V - E + F$
Cube	8	12	6	2
Octahedron	6	12	8	2
Tetrahedron	4	6	4	2
Egyptian Pyramid	5	8	5	2
Dodecahedron	20	30	12	2
Icosahedron	12	30	20	2
n-Cylinder	$2n$	$3n$	$n + 2$	2
n-Pyramid	$n + 1$	$2n$	$n + 1$	2
Truncated Icosahedron	60	90	32	2
Zome Ball	60	120	62	2

Zome Patterns Worksheet

Directions:

Important: When you finish with this exercise, please take any structures you have built completely apart and sort the pieces into the appropriate containers.

As a warm-up, use zome parts to build each of the following structures: a cube, an icosahedron and a dodecahedron. For each of these structures, you will need only blue struts, all having the same length. The struts are tough, but it is possible to break them. Push them straight into and pull them straight out of the Zome balls to make and break connections. The figure below shows models of the three structures you are trying to build.



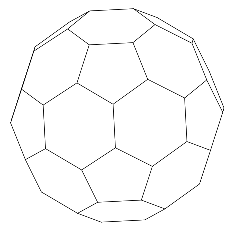
For each of the objects, count the number of vertices (Zome balls), edges (Zome struts) and faces (flat regions surrounded by balls and struts). We will use V, E and F to indicate the number of vertices, edges and faces, respectively of our structures. Try to figure out different ways to count them by dividing them into classes, or by whatever other means you can think of. These structures are relatively easy to count, but other examples will be more complex.

Whenever you build a structure and count V, E and F, enter that information into the table at the end of this worksheet.

Today we are interested in objects that do not contain "holes" in the sense that if you blow up a very flexible balloon inside the object, it would touch all the struts and balls without having to touch itself. (For example, a doughnut-shaped figure would not count, since if you inflated a balloon inside, two sections of the balloon's outer surface would eventually come in contact.) Another way to think of your objects is that if they were completely flexible, they could be distorted, without cutting, to the form of a sphere. A mathematician would say that they are topologically equivalent to a sphere.

Now build some more models that are "topologically equivalent to a sphere". Do not neglect simple models like tetrahedrons and octahedrons (which will require different-length and different-color struts). Other ideas include "cones", Egyptian pyramids, and so on. One easy way to make new models is to add faces to an existing model. For example, imagine adding a sort of Egyptian pyramid on top of a cube to make something that looks a bit like a house with a roof. For each of the models that you build, carefully count V, E and F, and record your results in the table at the end of the worksheet.

More challenging objects (both to build and to count the V, E and F features for include the “soccer ball” as illustrated below, and you can even make a large model of the Zome ball itself. The soccer ball will require only blue and yellow struts where all the blue struts are the same as are all the yellow ones, and the yellow strut size is as close as possible to the blue strut size. A model of the Zome ball itself will require two different lengths of blue struts. Again, count and record the V, E and F data for each structure.



After you have collected your data, try to find patterns in it. **Hint:** There is a simple algebraic expression relating V, E and F. Once you find the expression, try to come up with reasons why it might be true.

An advanced topic you can experiment with is to see if you can do a similar analysis of objects with a hole in them (objects that are topologically equivalent to a doughnut, or torus).

The cube and octahedron are said to be dual. So are the dodecahedron and icosahedron. The tetrahedron is said to be self-dual. What might this mean? **Hint:** Look at the values for V, E and F for these pairs of objects. Do other objects have duals in this sense?

Object	Vertices (V)	Edges (E)	Faces (F)	