Contest Geometry

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Abstract

This document contains a list of the more important formulas and theorems from plane Euclidean geometry that are most useful in math contests where the goal is computational results rather than proofs of theorems. None of the results herein will be proved, but it is a good exercise to try to prove them yourself.

1 Triangles

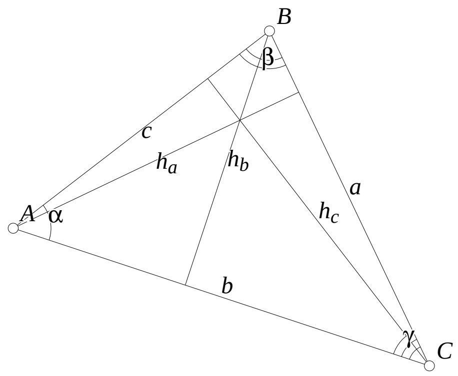

Figure 1: Generic Triangle

In what follows, we will use the triangle in Figure 1. In the figure, A, B, and C are the vertices; the angles at those vertices are α, β, and γ, respectively; the sides opposite them are a, b, and c, and the altitudes from them are h_a, h_b, and h_c.

1.1 Triangle Centers

In Figure 1 you can see that the three altitudes of the triangle seem to meet at a point. In fact they always do, as do the three medians and the three angle bisectors. The three altitudes meet at the orthocenter of the triangle, the three medians at the centroid, and the three angle bisectors at the incenter. The incenter is the center of the circle that can be inscribed in the triangle, and the centroid is the center of mass of the triangle (a triangle cut out of cardboard would balance at the centroid). The centroid divides each of the medians in a ratio of $2:1$.

If you construct the perpendicular bisectors of the sides of the triangle, they also meet at a point called the circumcenter which is the center of the circle that passes through the three vertices of the triangle.

The circumcenter, orthocenter, and centroid all lie on a straight line called the Euler line.

1.2 Area of a Triangle

There are many ways to calculate the area. If you know the base and the altitude (and keep in mind that *any* base and its corresponding altitude will do), then the area is given by:

$$\mathcal{A}(\triangle ABC) = \frac{a \cdot h_a}{2} = \frac{b \cdot h_b}{2} = \frac{c \cdot h_c}{2}.$$

If $s = (a+b+c)/2$ (s is called the "semiperimeter"), then here is Heron's formula for the area:

$$\mathcal{A}(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}.$$

If you know two sides and the included angle, here's the area:

$$\mathcal{A}(\triangle ABC) = \frac{ab\sin\gamma}{2} = \frac{bc\sin\alpha}{2} = \frac{ca\sin\beta}{2}.$$

1.3 Angles in a Triangle

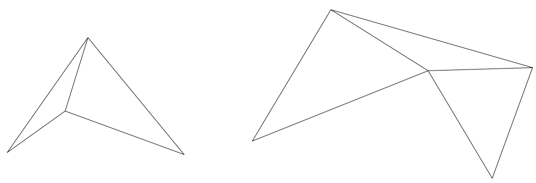

Figure 2: Sum of angles in a general polygon

The three angles of a triangle always add to $180°$, so in our general triangle in Figure 1, we have $\alpha+\beta+\gamma = 180°$. Everybody knows that, but remember that this lets you find the sum of the angles of a general simple quadrilateral, pentagon, hexagon, and so on. A "simple" polygon is one in which the sides do not cross each other, but the polygon need not be convex.

Notice in Figure 2 that the quadrilateral can be divided into two triangles, the pentagon into three, and so on. Thus, the sum of the interior angles of a simple quadrilateral is $2 \cdot 180°$, of a simple pentagon is $3 \cdot 180°$, and in general, of a simple n-sided polygon is $(n-2) \cdot 180°$.

1.4 The Pythagorean Theorem

In the special case where one of the angles in a triangle is a right angle, you can use the Pythagorean theorem to relate the lengths of the three sides. If angle A in the generic triangle is $90°$, then the Pythagorean theorem states that $a^2 = b^2 + c^2$. Note that by dropping an appropriate altitude, any triangle can be converted into a pair of right triangles, so in that sense, the theorem can be used on any triangle.

There are an infinite number of right triangles whose sides have integer lengths, the most common being the $3-4-5$ right triangle. It's probably worth memorizing a few of the smaller ones, including $5-12-13$, $7-24-25$, $8-15-17$ and $20-21-29$. Remember that any multiple is also a right triangle, so $6-8-10$ and $9-12-15$ are effectively larger versions of the $3-4-5$ right triangle.

If m and n are two integers, and $m > n > 0$, then the three numbers $m^2 - n^2$, $2mn$, and $m^2 + n^2$ form a Pythagorean triplet. All Pythagorean triplets are formed with a suitable choice of m and n. If one of the two numbers m and n is odd and the other is even, and if they are relatively prime, then the resulting Pythagorean triplet is primitive in the sense that it is not a multiple of a smaller triangle.

1.5 Other Special Triangles

Know the properties of the equilateral triangle, of the $45° - 45° - 90°$ triangle, and of the $30° - 60° - 90°$ triangle. The lengths of the sides are $1-1-1$, $1-1-\sqrt{2}$, and $1-\sqrt{3}-2$, respectively.

Another triangle that comes up in contests is the $13-14-15$ triangle that seems like an oddball at first, but it can be divided into two Pythagorean triangles: the $5-12-13$ and the $9-12-15$ triangles.

1.6 The Law of Sines and the Law of Cosines

For the generic triangle in Figure 1, here is the law of sines:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where R is the radius of the circumscribed circle.

The law of cosines is this:

$$\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos\alpha \\
b^2 &= c^2 + a^2 - 2ca\cos\beta \\
c^2 &= a^2 + b^2 - 2ab\cos\gamma.
\end{aligned}$$

Notice that if one of the angles is $90°$, the law of cosines is exactly the same as the Pythagorean theorem.

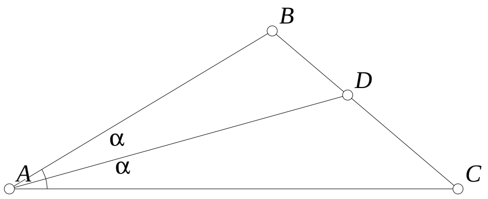

Figure 3: Angle Bisector Property

1.7 Angle Bisector Property

In Figure 3, if AD is the angle bisector of $\angle BAC$, then we have:

$$\frac{BD}{AB} = \frac{DC}{AC}.$$

1.8 Stewart's Theorem

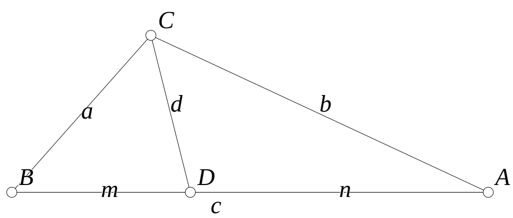

Figure 4: Stewart's Theorem

Stewart's theorem refers to any triangle $\triangle ABC$ with a line from one vertex to the opposite side as shown in Figure 4. With the sides labeled as in the figure, Stewart's theorem states that:

$$a^2n + b^2m = c(mn + d^2).$$

1.9 Menelaus' Theorem

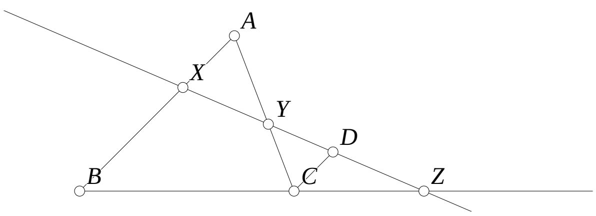

Figure 5: Menelaus' Theorem

Menelaus' Theorem refers to an arbitrary line cutting an arbitrary triangle, where the line may intersect the edges of the triangle either inside or outside the triangle. In fact, it may miss what you normally think of as the triangle altogether, but it will still hit the extensions of the lines. Obviously, it doesn't work if the line is parallel to one of the triangle edges. See Figure 5.

The theorem states that:

$$\frac{BZ}{CZ} \cdot \frac{CY}{AY} \cdot \frac{AX}{BX} = 1.$$

With a careful statement of the theorem with directed lengths of the sides Menelaus' theorem can predict whether the intersections are inside or outside the triangle.

1.10 Ceva's Theorem

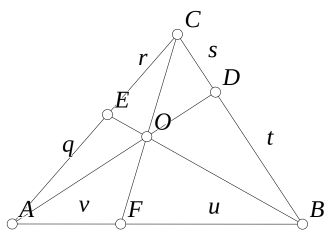

Figure 6: Ceva's Theorem

Ceva's theorem is similar to Menelaus' theorem in that it refers to an arbitrary triangle, and to an arbitrary point O. O can be inside or outside the triangle (but not on the edges). If lines are drawn through O and each of the vertices as in Figure 6, then:

$$\frac{AE}{CE} \cdot \frac{CD}{BD} \cdot \frac{BF}{AF} = 1,$$

or

$$usq = vtr.$$

As in Menelaus' theorem, if directed ratios are used, Ceva's theorem can be used to determine if the intersections of the lines AO, BO, and CO with the opposite sides are inside or outside the triangle, but Ceva's theorem with directed lengths will have the product above equal to -1 instead of 1.

2 Cyclic Quadrilaterals

Here are a few nice properties of quadrilaterals that can be inscribed in a circle as in Figure 7.

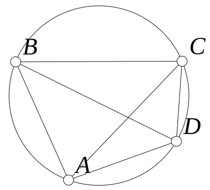

Figure 7: Cyclic Quadrilateral

2.1 Angles

If $ABCD$ is a cyclic quadrilateral, then

$$\angle DAB + \angle BCD = \angle ABC + \angle CDA = 180°.$$

Conversely, if $\angle DAB + \angle BCD = 180°$ or if $\angle ABC + \angle CDA = 180°$, then $ABCD$ is a cyclic quadrilateral.

2.2 Ptolemy's Theorem

For a cyclic quadrilateral as in Figure 7, we have:

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

2.3 Brahmagupta's Formula

If a cyclic quadrilateral has sides of lengths a, b, c, and d, and the semi-perimeter $s = (a + b + c + d)/2$, then the area is given by:

$$\mathcal{A}(ABCD) = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Notice the similarity of this formula to Heron's formula in Section 1.2. Set one of the lengths in Brahmagupta's formula to zero and get Heron's formula.

3 Circles

3.1 Area

The area of a circle is $A = \pi r^2$, where r is the radius of the circle and π is a constant, approximately equal to 3.14159265.

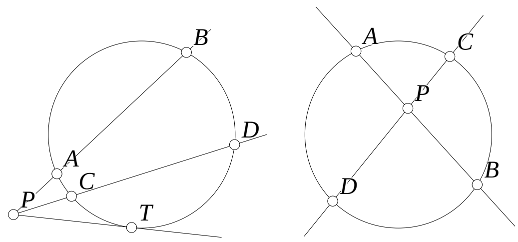

Figure 8: The Chord Theorem

3.2 The Chord Theorem

In both circles of Figure 8, we have:

$$PA \cdot PB = PC \cdot PD,$$

and if PT is tangent to the circle as in the leftmost circle of the figure, we have:

$$PA \cdot PB = PC \cdot PD = PT^2.$$

3.3 Circles and Angles

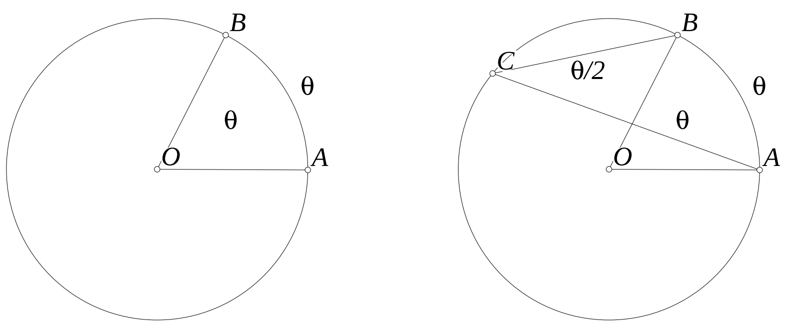

Figure 9: Central Angle and Inscribed Angle

The diagram on the left of Figure 9 gives the definiton of the measure of an arc of a circle. The arc from A to B is the same as the angle $\angle\theta$. In other words, if $\theta = 42°$, then the arc AB also has measure $42°$.

On the right of Figure 9 we see that an inscribed angle is half of the central angle. In other words, $\angle AOB = 2\angle ACB$.

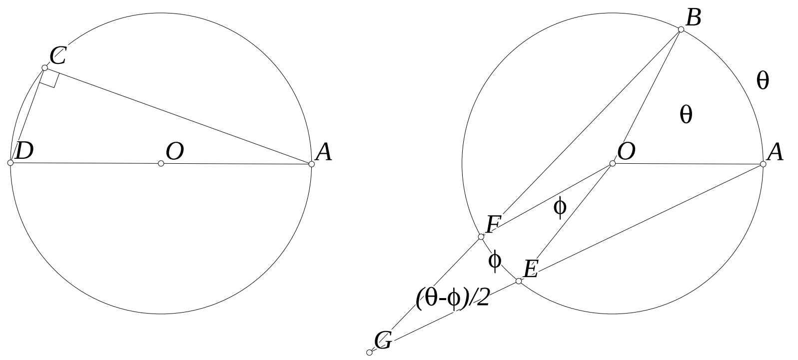

Figure 10: Right Angle, Difference of Angles

On the left of Figure 10 is illustrated the fact that an angle inscribed in a semicircle is a right angle. On the right of Figure 10 we see that an angle that cuts two different arcs of a circle has measure equal to half the difference of the arcs. In other words, $\angle AOB - \angle FOE = 2\angle EGF$.

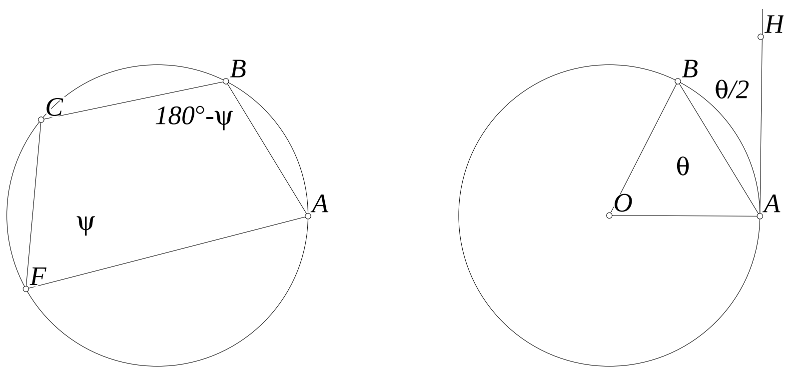

Figure 11: Inscribed Quadrilateral, Tangent

Finally, in Figure 11 we see that if a quadrilateral is inscribed in a circle, then the opposite angles add to 180°, and conversely, if the two opposite angles of a quadrilateral add to 180°, then the quadrilateral can be inscribed in a circle.

On the right, we see that a tangent line cuts off half of the inscribed angle: $\angle AOB = 2\angle HAB$.