Four Points on a Circle

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October 9, 2002

1 What is an "Interesting" Theorem?

What makes a Euclidean geometry theorem "interesting"? There are probably a lot of answers to this, but quite often it is when something occurs that is too lucky to be a coincidence. For example, if you draw some diagram and two of the points of the diagram lie on a line, that is not interesting at all, since *every* pair of points lie on some line. But if *three* points lie on a line when they were not defined to be on that line in the first place, the situation is usually interesting. In fact, if you randomly place three points on a finite portion of a Euclidean plane, selected in a uniform way, the probability that all three will lie on the same line is exactly zero.

Similarly, if two non-parallel lines meet at a point, that is not at all interesting, but if three or more do, it may be very interesting.

The same sort of idea applies to other figures. If two circles in the plane intersect in two points, or do not intersect at all, this is usually not interesting. But if they intersect at exactly one point (they are tangent, in other words), the situation is usually very interesting.

We begin this document with a short discussion of some tools that are useful concerning four points lying on a circle, and follow that with four problems that can be solved using those techniques. The solutions to those problems are presented at the end of the document.

2 Points on Circles

If two points lie on a circle, that is not interesting at all. In fact, for any two Euclidean points, there are an infinite number of circles that pass through both of them.

Three points on a circle is also not interesting. Unless the three points happen to lie on the same straight line (which has probability zero of occurring by chance), they lie on a circle. To see this, construct the perpendicular bisectors of the segments joining any two pairs of the points. Since the points do not lie in a line, the perpendicular bisectors are not parallel, and must meet somewhere. Any point on the perpendicular bisector of two points is equidistant from those points, so the point at the intersection of the two perpendicular bisectors is equidistant from all three points. Thus a circle centered at this intersection passing through any one of the three points must pass through the other two.

Since the first three points define a circle, it is almost impossible for the fourth point to lie on that circle by chance. Thus if, in a diagram, four points that are not in some sense defined to lie on a circle do lie on a circle, that is an interesting occurrence and it may indicate that an interesting theorem can be found.

Obviously, if five, six, or more points lie on a circle, that may be even more interesting, and one theorem we will examine here concerns the famous "nine point circle" where nine points related to any triangle all lie on the same circle. See Section 4.1.

3 Circle Preliminaries

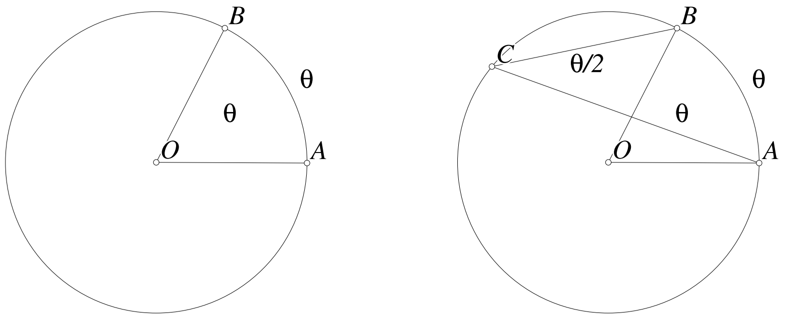

Figure 1: Central Angle and Inscribed Angle

There's basically one theorem we need to know to show a large number of interesting things about sets of points on a circle. Arcs of circles are measured by the central angle. In the left part of Figure 1, O is the center of the circle, and the measure of the arc AB is the same as the measure of the central angle. Both are called θ in the figure.

The right part of Figure 1 illustrates the theorem. If an angle is inscribed in the circle that intersects the circle in an arc of size θ, then the measure of the angle $\angle ACB$ is $\theta/2$. This is not obvious, and requires proof, but it is proved in every high school geometry course.

We must be a little careful here. In the figure, the two points A and B really determine two arcs—in this case a short one that's perhaps $50°$ in the figure, the rest of the circle (which is about $310°$ in the figure). If there is any question, we will name the two points that bound the arc in such a way that the arc is the part of the circle from the first to the second in a counter-clockwise direction. Thus in the figure, arc AB is about $50°$ and arc BA is about $310°$.

If, in Figure 1, the point C were on the arc AB (the small one), the angle $\angle ACB$ would measure half of about $310°$.

This basic fact about angles inscribed in a circle and the fact that a complete circle contains 360 degrees, allows us to prove a wide variety of interesting theorems.

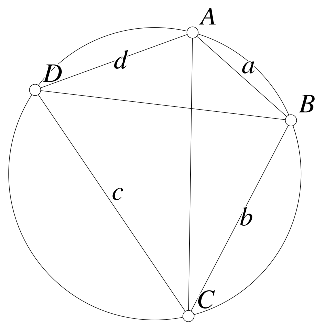

Figure 2: Cyclic Quadrilateral

Since we will be considering the case where four (or more) points lie on a circle, consider some easy (and not so easy) results displayed in Figure 2. In that figure, the four vertices A, B, C, and D lie on a circle in order. We will let a, b, c, and d represent the lengths of the segments AB, BC, CD, and DA, respectively.

$ABCD$ is a quadrilateral that is inscribed in a circle, and is called a cyclic quadrilateral (or sometimes a concyclic quadrilateral).

The most interesting and useful result is this: $\angle ABC + \angle CDA = 180°$ (and also, of course, $\angle DAB + \angle BCD = 180°$). This is easy to see, since $\angle ABC$ is half the measure of arc AC (measured counterclockwise) and $\angle CDA$ is half the measure of arc CA. But arcs AC and CA together make the entire circle, or $360°$, so the two opposite angles in any cyclic quadrilateral are supplementary, or in other words, add to make the straight angle.

It is not difficult to see the converse; namely, that if two opposite angles in a quadrilateral add to $180°$, then the quadrilateral is cyclic.

A very important special case of this is when the two opposite angles are right angles, or $90°$. This happens often, and when it does, the resulting quadrilateral is cyclic.

3.1 Ptolemy's Theorem

There is a very easy way to determine whether four points lie on a circle if you know the distances between them. Conversely, if you know that four points lie on any circle, the formula below, known as Ptolemy's Theorem, holds.

In Figure 2, if:

$$AB \cdot CD + BC \cdot DA = AC \cdot BD$$

then the four points A, B, C, and D all lie on a circle.

We will prove one direction of this result in Section 10.

3.2 Brahmagupta's Formula

Again, we will not prove this result here since it is not used in the problem set, but it also relates to cyclic quadrilaterals. The usual proof is straightforward, but involves a lot of patience with trigonometric manipulations that begin with the observation that the laws of sines and cosines apply to various triangles in the figure.

Brahmagupta's formula provides a method to calculate $\mathcal{A}(ABCD)$, the area of the cyclic quadrilateral $ABCD$. Here it is:

If $ABCD$ is a cyclic quadrilateral, whose sides have lengths a, b, c, and d, then if $s = (a+b+c+d)/2$ is the semiperimeter, its area is given by:

$$\mathcal{A}(ABCD) = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Notice that from Brahmagupta's formula it is trivial to deduce Heron's formula for the area of any triangle (remember that every triangle is concyclic). Here is Heron's formula for the area of a triangle $\triangle ABC$ having sides of lengths a, b, and c and semiperimeter $s = (a+b+c)/2$:

$$\mathcal{A}(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}.$$

The derivation of Heron's formula is obvious: pick a point D near C of $\triangle ABC$, but on its circumcircle. Then let D approach C. The term $(s-d)$ in Brahmagupta's formula will tend toward s, and the quadrilateral $ABCD$ will become more and more like the triangle $\triangle ABC$.

4 Problems

4.1 The Nine Point Circle

This is perhaps one of the most amazing theorems in Euclidean geometry. It states that if you begin with any triangle, the nine points consisting of the midpoints of the sides (points A', B', and C' in Figure 3), the feet of the altitudes (points E, F, and G), and the midpoints of the segments from the orthocenter to the vertices of the triangle (points Q, R, and S), all lie on the same circle.

(The orthocenter H is the intersection of the three altitudes.)

For the proof, see Section 5.

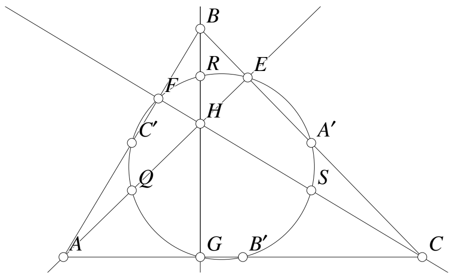

Figure 3: Nine Point Circle

4.2 Miquel's Theorem

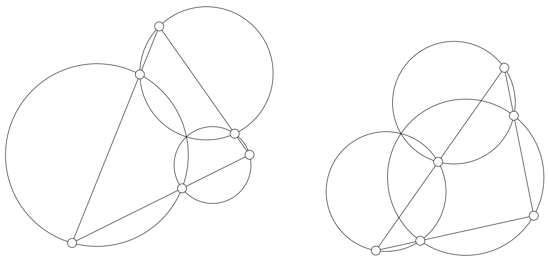

Figure 4: Miquel's Theorem

Figure 4 shows two examples of Miquel's theorem. Choose any triangle and then choose a point on each of its sides. Construct circles passing through each vertex of the triangle and through the points on the two adjacent sides. All three of those circles meet at a point.

The figure shows that the result seems to hold in two cases, but note that the point of intersection of the three circles may not lie within the original triangle.

In fact, the theorem even holds if the points on the sides of the triangle do not lie between the vertices of the triangle.

For the proof, see Section 6

4.3 Napoleon's Theorem

Napoleon's Theorem (see Figure 5) says that given any triangle, if you erect equilateral triangles on the edges of that triangle, all pointing outward, that the centers of those three triangles will form an equilateral triangle. In the figure three circles are drawn to give a hint as to how the proof should proceed. Notice that those three circles all seem to meet at a point.

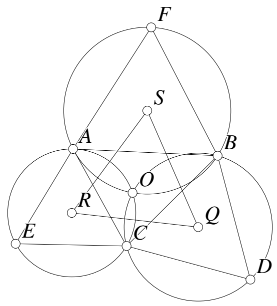

Figure 5: Napoleon's Theorem

For the proof, see Section 7

4.4 The Simson Line

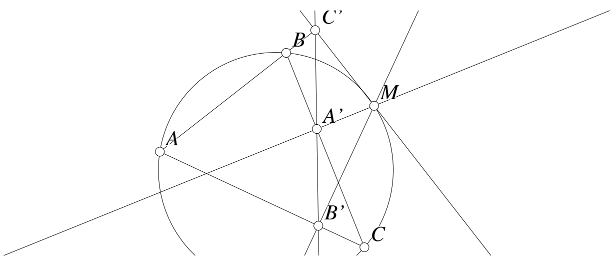

Figure 6: The Simson Line

Let M be any point on the circumcircle of $\triangle ABC$. From M, drop perpendiculars to each of the sides of the triangle. Show that the points of intersection of the perpendiculars with the sides all lie on a line called the Simson Line. See Figure 6

For the proof, see Section 8

4.5 Fagnano's Problem

Fagnano's problem is the following: Given an acute-angled triangle $\triangle ABC$, find the triangle with the smallest perimeter whose vertices lie on the edges of $\triangle ABC$. In Figure 7, this triangle with smallest perimeter is $\triangle DEF$.

The answer is that it is the triangle whose vertices are the feet of the altitudes of the original triangle. The easiest proof uses some interesting arguments involving not only circles, but reflections of the original triangle.

For the proof, see Section 9

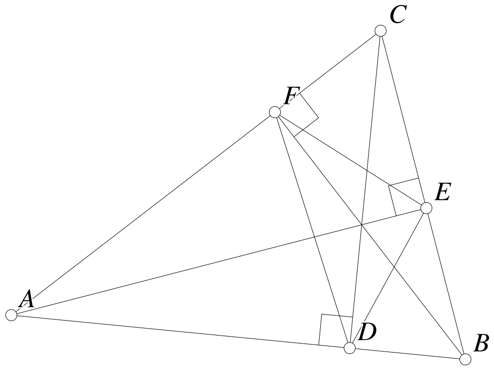

Figure 7: Fagnano's Problem

5 Proof: Nine Point Circle

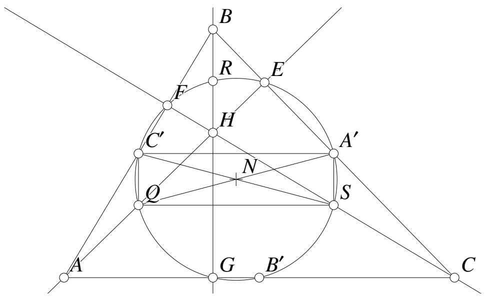

Figure 8: Nine Point Circle

See Figure 8. We will prove that all nine points lie on the circle by first showing that the six points A', C', Q, S, E and F all lie on a circle. The proof will use the line AC as the base of the triangle. But every triangle has three bases, and if we consider the line AB to the the base, exactly the same proof would show that the points A', B', R, Q, E, and G also lie on the same circle. Since these two circles have three points in common: A', Q, and E, they must be the same circle, and hence all nine points lie on the same circle.

Since A' and C' are midpoints of the sides of $\triangle ABC$, we know that $A'C' \parallel AC$. Similarly, since Q and S are midpoints of the sides of $\triangle AHC$, $QS \parallel AC$. But since both QS and $A'C'$ are parallel to the same line AC, they are parallel to each other: $QS \parallel A'C'$.

Using the same reasoning, since A' and S are midpoints of BC and HC in $\triangle BHC$, $A'S \parallel BH$. If we look at $\triangle BHA$, repeat the reasoning to show that $C'Q \parallel BH$.

Since $C'Q$ and $A'S$ are both parallel to the same line BH, they are parallel to each other. Thus $A'C'QS$ is a parallelogram.

But it is more than a parallelogram; it is a rectangle. Since BH is part of the altitude of the triangle $BH \perp AC$. Because $A'S$ and QS are parallel to perpendicular lines, $A'S \perp QS$. Therefore $A'C'QS$ is a rectangle.

Let N be the center of the rectangle. Obviously N is equidistant from A', C', Q, and S, so all four of those points lie on a circle centered at N.

In this circle, $A'Q$ and $C'S$ are diameters. Since the lines QE and SF are altitudes, $\angle A'EQ = \angle C'FS = 90°$. But these right angles subtend the diameters $A'Q$ and $C'S$, so E and F lie on the circle centered at N and passing through points A', C', Q, and S. Thus we have shown that six of the nine points lie on a circle.

As we said above, there is no reason to assume that AC is the only base, and using either AB or BC as base we can show that other sets of six points lie on circles, and since each set has three points in common with the two other sets, all nine points lie on the same circle.

6 Proof: Miquel's Theorem

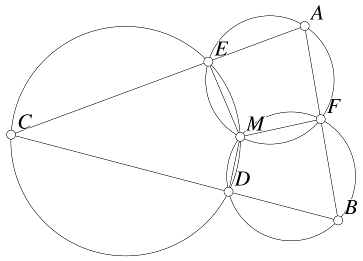

Figure 9: Miquel's Theorem

The proof is fairly simple; see Figure 9.

Using the labels in Figure 9, begin by considering only two of the circles—AEF and BFD. They will meet at some point M, and our goal will be to show that the point M also lies on the circle CED.

Construct lines DM, EM, and FM, forming two cyclic quadrilaterals $AEMF$ and $BDMF$. Since the quadrilaterals are cyclic, we know that $\angle EAF + \angle EMF = 180°$ and that $\angle FBD + \angle DMF = 180°$. We also know that $\angle EAF + \angle FBD + \angle DCD = 180°$ since they are the three angles of a triangle, and we know that $\angle EMF + \angle DMF + \angle EMD = 360°$.

Apply a little algebra to those equations to conclude that $\angle DCE + \angle EMD = 180°$. If two opposite angles in a quadrilateral add to $180°$, we know that it is a cyclic quadrilateral, so the points C, E, M and D all lie on a circle. Thus the three circles meet at the point M.

7 Proof: Napoleon's Theorem

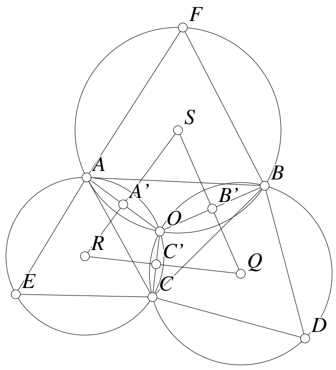

Figure 10: Napoleon's Theorem

The proof of Napoleon's Theorem is similar in some ways to the previous proof. See Figure 10. We begin by constructing circles around the three equilateral triangles and we will show that they all meet at a point.

In the figure, consider the two circles ABF and CBD. In addition to meeting at B they will meet at another point O. Since quadrilaterals $AOBF$ and $BOCD$ are cyclic, we know that opposite angles in them add to 180°. Thus $\angle AOB + \angle BFA = \angle COB + \angle BDC = 180°$. But the triangles $\triangle ABF$ and $\triangle CBD$ are equilateral so $\angle BFA = \angle BDC = 60°$. Thus we can conclude that $\angle COB = \angle AOB = 120°$.

Since the angles $\angle COB$, $\angle AOB$, and $\angle COA$ add to make a full 360°, $\angle COA = 120°$. But then since $\angle COA + \angle CEA = 180°$, quadrilateral $EAOC$ is cyclic, so the three circles surrounding the exterior equilateral triangles meet in a single point O. Not only that, but the lines AO, BO, and CO all meet at 120° angles.

The points Q, R, and S are centers both of the equilateral triangles and of the circles, so the lines connecting them are perpendicular to the lines AO, BO, and CO. Thus $\angle SA'O = \angle SB'O = 90°$ and since they add to 180°, quadrilateral $SA'OB'$ is cyclic and $\angle A'SB' + 120° = 180°$. Therefore $\angle A'SB' = 60°$. But the same argument can be made to show that $\angle C'QB' = \angle A'RC' = 60°$, so $\triangle QRS$ is equilateral.

8 Proof: The Simson Line

The following proof works only if M lies on the arc between B and C. If it is on one of the other arcs, rename the points appropriately before beginning.

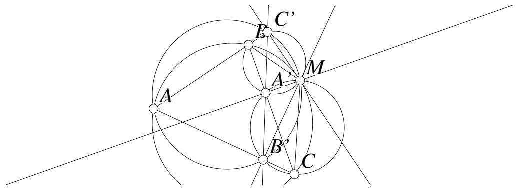

Figure 11: The Simson Line

Since the lines dropped from M are perpendicular to the sides of the triangle, they form right angles, and it is easy to see in Figure 11 that $MA'BC'$, $MA'B'C$ and $MC'AB'$ are sets of concyclic points. Obviously $ABMC$ is also concyclic, since M is on the circumcircle of $\triangle ABC$.

Since $ABMC$ and $MC'AB'$ are concyclic sets of points, both $\angle BMC$ and $\angle C'MB'$ are supplementary to $\angle BAC$. Therefore they are equal. Since $\angle BMC = \angle BMB' + \angle B'MC$ and $\angle C'MB' = \angle BMB' + \angle C'MB$ we find that $\angle B'MC = \angle C'MB$. But since each of those angles lies in a circle, $\angle C'MB = \angle C'A'B$ and $\angle B'MC = \angle B'A'C$.

The equality of the final pair of angles proves that the points B', A' and C' are collinear, since equal vertical angles are guaranteed at A'.

9 Proof: Fagnano's Problem

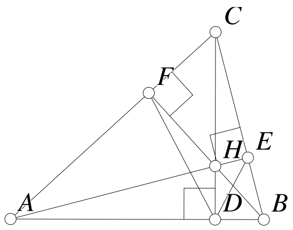

Figure 12: Equal Angles

To prove that the triangle connecting the feet of the altitudes of an acute triangle has the minimum perimeter, we first need to prove a lemma. We will show that this "pedal" triangle $\triangle DEF$ in Figure 12 makes equal angles with the bases of the triangles. In other words, we need to show that $\angle BDE = \angle ADF$.

We begin by considering the two circles passing through AFD and BED. It is clear that both pass through H, the orthocenter of $\triangle ABC$ since both quadrilaterals $ADHB$ and $BDHE$ contain two opposite right angles and are hence concyclic.

But $\angle FHA = \angle EHB$ since they are vertical angles, and since the two quadrilaterals $ADHB$ and $BDHE$ are concyclic, $\angle BDE = \angle EHB$ and $\angle ADF = \angle FHA$ since they subtend equal arcs of the circles. Thus $\angle BDE = \angle ADF$.

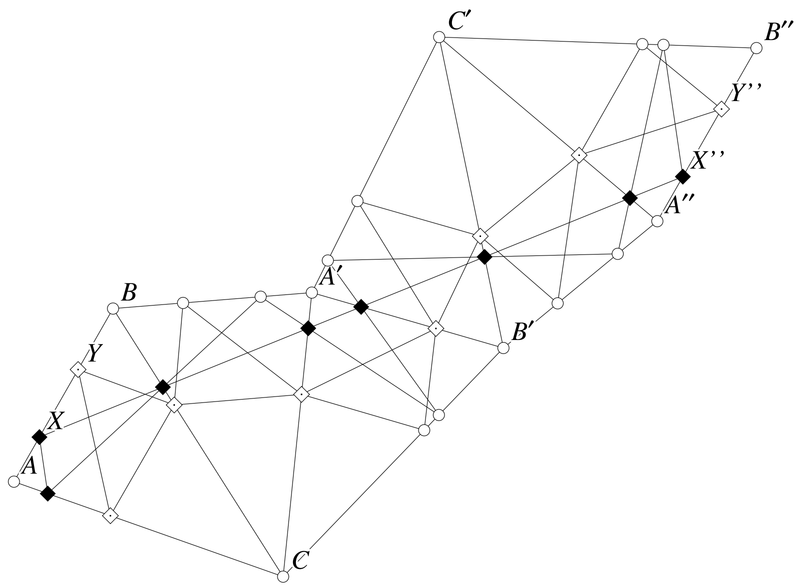

Figure 13: Fagnano's Problem

With the lemma above, and with a somewhat miraculous construction, the proof of the main theorem is not difficult. See Figure 13.

Reflect the original triangle $\triangle ABC$ across the line BC forming a triangle $\triangle A'BC$. Reflect this one across $A'C$ forming another triangle $\triangle A'B'C$. Continue in this way for five reflections, as in the figure.

Chase angles to show the line $B''A''$ is parallel to BA. It is easy to show the reflections of the triangle connecting the feet of the altitudes (the one with solid diamond vertices) forms a straight line after the reflections. Any other triangle (like the one in the figure with dotted diamond vertices) will not be reflected to form a straight path.

The beginning and end points of these two paths (X to X'' and Y to Y'') are equally far apart, since they form a parallelogram. It is a parallelogram since $B''A''$ is parallel to BA and the length XY is the same as the length $X''Y''$. But the solid-diamond path is a straight line, so any other path must be longer.

Note that the two paths each consist of two copies of the sides of the pedal triangles, so clearly the pedal triangle connecting the bases of the altitudes is the shortest.

10 Proof: Ptolemy's Theorem

The statement of Ptolemy's theorem contains a bunch of products of lengths of segments:

$$AB \cdot CD + BC \cdot DA = AC \cdot BD. \tag{1}$$

Generally, to prove theorems like this we need to convert those products into ratios, and then use similar triangles or some other technique to establish the ratios.

It turns out not to matter much how we start, but notice that in equation 1 if we could get the ratio AB/AC that might help. But AB and AC are not in any pair of similar triangles, so let's just construct a line that *creates* a pair of similar triangles with the side AB in one corresponding to side AC in the other.

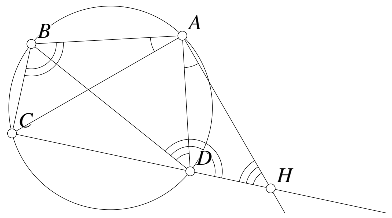

Figure 14: Ptolemy's Theorem

In Figure 14, construct the line from A outside the cyclic quadrilateral that makes the same angle with AD that AC does with AB and intersects the line CD at H. Now we have the similarity that we want. $\angle BAD = \angle CAH$ since both are equal to $\angle CAD$ plus $\angle BAC$ or $\angle DAH$ which were constructed to be equal. Since $\angle ABD$ and $\angle ACD$ are inscribed in the same circle, they are also equal, so by angle-angle similarity, we know that $\triangle ABD \sim \triangle ACH$.

But there's another pair of similar triangles. Since $\angle ABC$ and $\angle CDA$ are opposite angles in a cyclic quadrilateral, they are supplementary. It is also obvious that $\angle CDA$ and $\angle HDA$ are supplementary, so $\angle ABC = \angle HDA$. Since $\angle HAD = \angle CAB$ (by construction), we have, again by angle-angle similarity, that $\triangle ABC \sim \triangle ADH$.

Finally, notice that

$$CH = CD + DH. \tag{2}$$

From the similarities of the pairs of triangles, we have:

$$\frac{AC}{AB} = \frac{CH}{BD} \quad \text{and} \frac{\text{AD}}{\text{AB}} = \frac{\text{DH}}{\text{BC}}. \tag{3}$$

If we solve for DH and CH in equations 3 and substitute them into equation 2, we obtain:

$$\frac{AC \cdot BD}{AB} = CD + \frac{AD \cdot BC}{AB}.$$

When we multiply through by AB we obtain Ptolemy's theorem.