Euler's Theorem

Tom Davis
tomrdavis@earthlink.net
http://www.geometer.org/mathcircles
July 14, 2009

Abstract

Euler's theorem is a nice result that is easy to investigate with simple models from Euclidean geometry, although it is really a topological theorem. One of the advantages of studying it as presented here is that it provides the student many exercises in mental visualization and counting.

1 The Cube

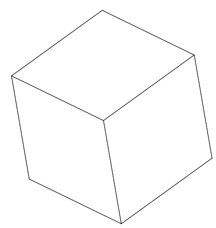

Figure 1: Cube

We will begin not with the simplest, but with what is probably the most familiar example of a polyhedron for most students: the cube (see Figure 1). Here, as well as in most of the examples in this article, we want to count the number of faces, edges, and vertices in the polyhedron. The faces are the flat regions, of which a cube has 6. The edges are the line segments between pairs of faces (a cube has 12) and the vertices are the "corners" or the points of intersection of the edges, and a cube has 8 of those.

Surprisingly, even the cube is complex enough that some people may get these numbers wrong, so it is worthwhile to count the features carefully, and especially to think of counting methods that guarantee a complete and accurate count. As always in counting, it is best if you can count the same things in different ways since this provides a check of your answer.

It is useful to have physical models of some of the objects used as examples here, especially at first, when you are getting used to counting the features. It is, however, a wonderful visualization exercise to try to make these counts in your head. In Appendix B there are cutouts that can be used to construct most of the models we consider here.

To count the faces of a cube, orient it so that you are looking at a face, and so that two of the faces are parallel to the floor. There are three kinds of faces: top/bottom, left/right, and near/far, making a total of 6.

To count the vertices, there are two sets: four around the top face and four around the bottom, for a total of 8.

Counting edges is the most difficult, and there are a few ways to do it. Notice that there are 4 around the top, 4 around the bottom, and 4 connecting the top and bottom. Alternatively, there are 4 perpendicular to the floor, 4 whose extensions would pass though your body, and 4 that go right and left.

The results for the cube is this: if F, E and V represent the number of faces, edges, and vertices, we have: $F = 6$, $E = 12$ and $V = 8$. On a separate piece of paper (or off to the side of the blackboard if you're presenting this to a class), begin to make a table that looks like this with blank space available for a few additional entries. Leave space for one additional column on the right:

Object	Faces (F)	Edges (E)	Vertices (V)	
Cube	6	12	8	

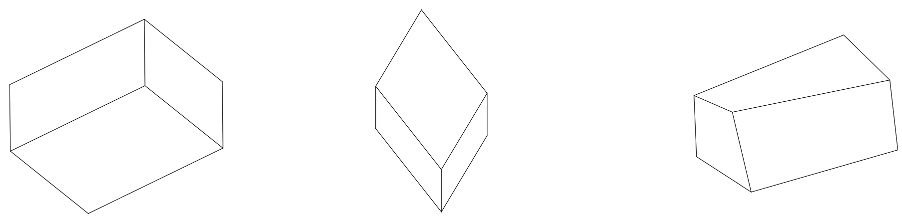

Figure 2: Two Parallelepipeds and a Distorted Cube

Before continuing, it is useful to make a couple more observations. First, note that there is nothing special for our purposes about the normal cube in Figure 1 where all the sides and angles are the same. Figure 2 displays three other examples that are simply distortions of the cube, but which contain the same number of faces, edges, and vertices. In fact, in the rightmost example of that figure, the faces are not even flat.

Second, here is another method to count the features. For the cube, this method seems a bit clumsy, but for some of the solids we will examine later, this method can vastly simplify the counting.

Suppose we know that the cube has 6 faces. Imagine a model of the cube made of paper, and with a pair of scissors, cut the cube into pieces so that each of the original faces is now a square of paper. After cutting, there are 6 squares, each of which has 4 edges, so there are $6 \times 4 = 24$ edges in all. But notice that each time you cut a single edge on the original cube, it became two edges in the collection of cut squares: one on each of the two squares that were originally joined at that edge. Thus, your count of 24 edges includes each original edge twice, so the original edge count must have been $24/2 = 12$ edges.

Count the vertices similarly. After the cube is cut into 6 squares, each has 4 vertices for a total of $6 \times 4 = 24$. In the original cube each vertex before cutting will lie on the corners of three squares. Thus each original vertex appears three times in the cut version, so the original number of vertices was $24/3 = 8$.

2 The Tetrahedron

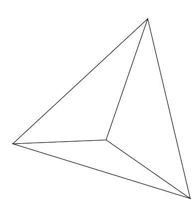

Figure 3: Tetrahedron

The simplest polyhedron is the tetrahedron illustrated in Figure 3. The only reason we considered the cube first is that the cube is far more familiar to most people.

Inspection of the tetrahedron gives $F = 4$, $E = 6$, and $V = 4$, but it is worthwhile to find alternative counting methods. (The name "tetrahedron" itself gives away one of the answers—"tetra" means "four" and “hedron” means “faced”.)

With one triangular base parallel to the floor and the tip up, we can count the vertices around the base (3) plus the tip to get $V = 4$. A similar argument counts the faces. There are 3 edges around the base, and 3 connecting the tip to the base for a total of 6.

Or cut the tetrahedron into 4 triangles as you did with the cube: The cut triangles contain $4 \times 3 = 12$ vertices and 12 edges, but each edge is counted twice and each vertex three times, giving $E = 12/2 = 6$ and $V = 12/3 = 4$.

Be sure to add the tetrahedron to your table, which should now look like this:

Object	Faces (F)	Edges (E)	Vertices (V)	
Cube	6	12	8	
Tetrahedron	4	6	4	

3 An Egyptian Pyramid

Everyone can visualize an Egyptian pyramid. Try to count the edges, faces, and vertices mentally. Here, we are only concerned with closed solids, so be sure to count the square bottom as an additional face. If your visualization is weak, a drawing appears in Appendix A, Figure 24.

Be certain to add your results to the table. (The completed table at the end of Section 4 contains the pyramid data so you can check your work.)

4 The Octahedron

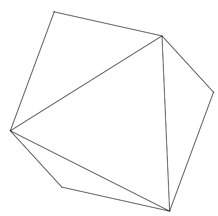

Figure 4: Octahedron

The name “octahedron” (“eight-faced”) gives away one of the answers immediately. The octahedron is the 8-faced polyhedron (“polyhedron” = “many-faced”) illustrated in Figure 4. It looks like two Egyptian pyramids with their bases glued together.

To count the features, hold the octahedron with a vertex up. There are obviously 4 triangles in the top half and 4 in the bottom for a total of 8. There are 4 vertices around the middle as well as one at the top and bottom for a total of 6. Finally, there are 4 edges around the middle plus 4 connecting to the top and 4 more connecting to the bottom for a total of 12.

Alternatively, cut the figure into 8 triangles having a total of $8 \times 3 = 24$ edges and 24 vertices. The edges are double counted, and the vertices are counted 4 times each, so there are $24/2 = 12$ edges and $24/4 = 6$ vertices[1].

Our table of values now looks like this:

[1] Or, if you’ve worked out the numbers for an Egyptian pyramid ($F = 5$, $E = 8$ and $V = 5$), you can double those when you glue two together, but 4 edges, 4 vertices, and 2 faces disappear since the edges and vertices merge and the two faces become interior. This gives: $F = 2 \times 5 - 2 = 8$, $E = 2 \times 8 - 4 = 12$ and $V = 2 \times 5 - 4 = 6$.

Object	Faces (F)	Edges (E)	Vertices (V)	
Cube	6	12	8	
Tetrahedron	4	6	4	
Egyptian Pyramid	5	8	5	
Octahedron	8	12	6	

4.1 Duality

The octahedron is the dual of a cube. The dual of a convex polyhedron is obtained by placing a vertex of the new (dual) polyhedron in the center of each face of the original polyhedron and if two faces of the original polyhedron are joined by an edge, then the vertices in the centers of those faces are joined by an edge in the dual. Visualize a point in the center of each of the 6 faces of a cube connected as described above, and you can see that the result will be an octahedron.

A property of dual polyhedra is that the dual of the dual is the original polyhedron. The dual of the octahedron is the cube. By the construction of the dual each face in the original corresponds to a vertex in the dual (the vertex placed in the center of the face), and each vertex in the original corresponds to a face in the dual. (Why is this?) The number of edges in a polyhedron and its dual are always equal, since a new edge is constructed for every edge connecting two faces in the original, so the counts for the dual are the same as the counts in the original, but with the number of faces and vertices interchanged.

Visualize the dual of a tetrahedron. Did you get a tetrahedron? What is the Egyptian pyramid's dual?

5 The Dodecahedron and Icosahedron

Figure 5: Dodecahedron/Icosahedron

Most people know that "tetra" means "four" and "octa" means "eight", but fewer know that "dodeca" means "twelve" and "icosa" means "twenty". If you do, it's easy to guess the number of faces of the dodecahedron and the icosahedron, illustrated in Figure 5.

5.1 Dodecahedron

Here is the first case where counting the features is a bit difficult. Hold the dodecahedron with one face parallel to the floor, and its opposite face will also be parallel to the floor. From the top face, 5 other faces hang down and from the bottom, 5 others "hang" up, making 12 total faces, as the "dodeca" tells us there should be.

Counting vertices and edges can also be done from this configuration. There are 5 vertices around the top and bottom, and 5 each at the bottoms of the edges coming down from the top or up from the bottom for a total of 20 vertices.

There are 5 edges around the top and bottom, 5 going down from the top, 5 coming up from the bottom, and the 10 endpoints of those down-going or up-going edges are connected in a loop with 10 more edges for a total of $5 + 5 + 5 + 5 + 10 = 30$ edges.

The analysis "by scissors" is easier: Cut the figure into 12 pentagons and you will have $12 \times 5 = 60$ edges and 60 vertices. After cutting, the edges in the original are each counted twice and the vertices 3 times, giving $60/2 = 30$ edges and $60/3 = 20$ vertices.

Find other ways to count the faces, edges, and vertices of the dodecahedron.

5.2 Icosahedron

If you notice that the icosahedron is the dual of the dodecahedron, you get $F = 20$, $E = 30$ and $V = 12$. (Remember that we simply need to reverse the vertex and face counts in the dual.)

To do the scissors analysis, you first need to count the faces. Orient the icosahedron with one vertex up and its opposite down. There are 5 faces touching the top and bottom vertices, as well as a ring of 10 around the middle (or visualize one set of 5 pointed "teeth" going down and another 5 pointing up). Thus there are 20 total faces.

After cutting up the icosahedron with scissors, there are $20 \times 3 = 60$ vertices and 60 edges in the resulting triangles, where each edge is double counted and each vertex is counted 5 times. Thus there are $60/2 = 30$ edges and $60/5 = 12$ vertices.

Try to count the number of vertices and edges directly.

In any case, if we add this data to our table, we obtain:

Object	Faces (F)	Edges (E)	Vertices (V)	
Cube	6	12	8	
Tetrahedron	4	6	4	
Egyptian Pyramid	5	8	5	
Octahedron	8	12	6	
Dodecahedron	12	30	20	
Octahedron	20	30	12	

Before reading on, take a look at all the data in this table and see if you can find any numerical relationships among the numbers in each row.

6 Euler's Theorem

There is enough numerical information in the table at the end of the previous section to find an interesting relationship among the numbers F, E and V. If you have not yet done so, please try to find that relationship yourself. (Hint: it is a linear relationship.)

Another hint is this: Because every one of our figures has a dual figure, the values of F and V must enter the relationship in the same way. In other words, any equation you obtain connecting the values of F, E and V must remain the same if F and V are interchanged so an equation like $F + 3V - 5E = 11$ cannot hold since interchanging F and V would give $V + 3F - 5E = 11$.

With these hints and a little playing around, you should obtain the equation known as Euler's theorem:

$$F - E + V = 2. \tag{1}$$

Obviously we do not yet have a proof (or even a complete statement of the theorem—are there any conditions the polyhedron must satisfy to guarantee the truth of the theorem?) We simply have counted the number of faces, edges, and vertices of 6 polyhedra and noticed that the relationship displayed in Equation 1 holds for all of them.

Although the scientific method can never prove anything, it can certainly be applied to mathematics to increase our confidence in the truth of a proposed theorem. Now that we think we have a formula relating F, E, and V, let us verify that it holds in at least a few other cases.

6.1 Cylinders (or Prisms)

Figure 6: Pentagonal and Octagonal Cylinders

Figure 6 illustrates both a pentagonal and an octagonal cylinder. They are constructed by "extruding" a base that is either a pentagon or an octagon to make a solid.

The features of both are easy to count. For the pentagonal cylinder, there are 5 faces around the outside and 2 on the top and bottom for a total of 7. There are $5 \times 2 = 10$ edges around the top and bottom and 5 more connecting the top and bottom for a total of 15. Finally, there are 5 vertices on both top and bottom for a total of 10. Thus $F = 7$, $E = 15$ and $V = 10$. $F - E + V = 7 - 15 + 10 = 2$, so our proposed theorem still seems to hold.

Similar reasoning for the octagonal cylinder gives $F = 10$, $E = 24$ and $V = 16$. $F - E + V = 10 - 24 + 16 = 2$, giving one more data point.

Why not add an infinite number of data points? Let's work out the numbers for a cylinder with n sides. The reasoning is identical: There are $n + 2$ faces, $3n$ edges, and $2n$ vertices. $F - E + V = (n + 2) - 3n + 2n = 3n + 2 - 3n = 2$.

What is the dual of an n-sided cylinder?

6.2 General Pyramids (or Cones)

Figure 7: Generalized Pyramids

We solved this for an Egyptian pyramid with a square base, but we can construct pyramids with any kind of polygonal base. See Figure 7 for examples of pyramids with 5-sided and 15-sided bases.

The analysis is similar to what we did in Section 6.1. It is a good exercise to do it yourself.

As above, it is easy to count the features for an infinite number of cases. Consider the situation where our pyramid has an n-sided base. There are $n + 1$ faces (the base plus the n triangular faces), there are $2n$ edges—n around the base and n more connecting the base to the tip, and there are $n + 1$ vertices—n around the base plus the tip. Thus $F = n + 1$, $E = 2n$ and $V = n + 1$. Euler's theorem continues to hold: $F - E + V = (n + 1) - 2n + (n + 1) = 2$.

What is the dual of an n-sided pyramid?

7 The Truncated Icosahedron

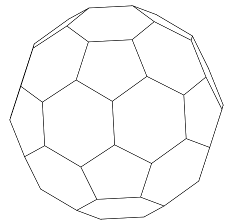

Figure 8: Truncated Icosahedron (or Soccer Ball)

We will delay the proof of Euler's theorem for one more section, but if you wish, you can jump ahead. This example is more difficult, and although it does give one more data point, it is probably most useful as an advanced exercise in counting. The mathematical name of the shape is a truncated icosahedron (an icosahedron with the corners cut off), but it looks a lot like the patterns on a soccer ball. See Figure 8.

Even counting the faces is difficult, but (particularly if you are holding a model in your hands) it is not too hard to see that there are 12 pentagonal faces. There is one on top and one on the bottom, and if you follow the points down from the top or up from the bottom, in every case you will arrive at another pentagon. Thus there are 6 pentagons in the top half and another 6 on the bottom.

If you know it is an icosahedron with the vertices chopped off, each vertex chopped gives us a pentagon, and there were none to begin with. An icosahedron has 12 vertices, so there must be 12 pentagonal faces.

To count the hexagons, notice that each edge of each pentagon is shared with exactly one hexagon. But each hexagon touches only 3 pentagons, so the $12 \times 5 = 60$ pentagon edges are used three at a time on the hexagons, making $60/3 = 20$ hexagons.

Alternatively, there were originally 20 faces on our icosahedron before we started whacking off the corners, and none were eliminated, so 20 remain afterwards, but the cutting converts triangles to hexagons.

The final count is 12 pentagons plus 20 hexagons for a total of $F = 32$.

Now use the scissors. After cutting, there will be 12×5 plus 20×6 or $60 + 120 = 180$ edges and 180 vertices among the cut pieces. Each edge in the original object appears on two pieces so there are $180/2 = 90$ edges. As you can see in Figure 8, every vertex marks the intersection of exactly 3 faces, so there are $180/3 = 60$ vertices. So $F = 32$, $E = 90$ and $V = 60$, so again $F - E + V = 32 - 90 + 60 = 2$.

For completeness, here is the final form of our table. If you're presenting it in class, note that a final column has been filled to illustrate the sum $F - E + V$:

Object	Faces (F)	Edges (E)	Vertices (V)	$F - E + V$
Cube	6	12	8	2
Tetrahedron	4	6	4	2
Egyptian Pyramid	5	8	5	2
Octahedron	8	12	6	2
Dodecahedron	12	30	20	2
n-Cylinder	$n + 2$	$3n$	$2n$	2
n-Pyramid	$n + 1$	$2n$	$n + 1$	2
Truncated Icosahedron	32	90	60	2

8 A Proof of Euler's Theorem

Before we prove the theorem, we had better state exactly what it says:

Theorem 1 (Euler's Theorem) *The number of faces, F, edges, E, and vertices, V, of a simple polyhe-*

dron are related by the formula $F - E + V = 2$.

The term “simple polyhedron” in the statement of the theorem above means a polyhedron that is in one piece without holes. Obviously a polyhedron that consisted of two unconnected cubes would satisfy $F - E + V = 4$ since you would just double all the counts, but two unconnected cubes do not constitute a simple polyhedron.

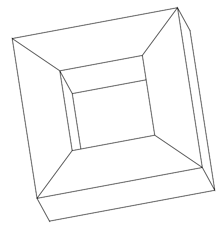

Figure 9: Box with Hole (Donut?)

Similarly, the object illustrated in Figure 9 is not a simple polyhedron since there is a hole passing through it. In this case it is not hard to determine that $F = 16$, $E = 32$ and $V = 16$, so $F - E + V = 0$[2].

The proof of Euler’s theorem is topological. Imagine that the polyhedra are not made of paper, but of rubber that we can stretch as much as we want.

Figure 10: Stretched Cube (2 Versions)

Take any polyhedron and punch a hole in the middle of one of its faces. Now grab the edges of that hole and pull and pull and pull until the hole is much bigger than the original piece of rubber. Doing this to the cube, for example, will result in a figure something like that displayed in Figure 10. The vertices of the original cube are displayed as tiny circles, and the large circle surrounding everything represents the enormously stretched hole in one of the faces. For a cube, it is easy to draw the stretched version where all the edges remain straight lines as in the example on the left. It makes no difference to our argument, however, if they are curved as illustrated by the example on the right.

The vertices and edges remain vertices and edges. When we count the faces, we have to be sure to include the one on the “outside”, since that is the one in which we initially punched a hole. In any case, check that in the stretched, flattened cube, there are 6 faces (including the outside one), 8 vertices, and 12 edges.

Draw a few more flattened versions of the other polyhedra we have examined earlier in this paper and then count the features and make certain they agree which our previous counts. In fact, a better definition of a “simple polyhedron” is simply any polyhedron that can be stretched out in this way to make a planar diagram. Try to convince yourself that the box with a hole in it shown in Figure 9 *cannot* be stretched out from a single hole in any of its faces.

[2]It turns out that an extended form of Euler’s theorem holds (that we will not prove here) which states that for a polyhedron with n “holes”, the correct formula is $F - E + V = 2 - 2n$.

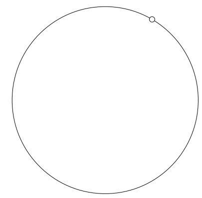

Figure 11: Simple "Polyhedron"

The key to the proof is this. For *any* such figure, we can begin with the very simple diagram in Figure 11 and convert it to the figure we desire simply by adding vertices to existing edge segments or by adding edge segments connecting two existing vertices.

The simple polyhedron consists of one vertex and one edge connecting it to itself. We can see that $F = 2$ (remember the outside face), and $V = E = 1$, so $F - E + V = 2$. If we add a vertex to an existing edge, both E and V are increased by one so $F - E + V$ remains unchanged. Similarly, adding an edge connecting any two existing vertices adds one edge and divides an existing face into two parts, thus adding one more face. If both F and E are increased by one, again we find that $F - E + V$ remains unchanged.

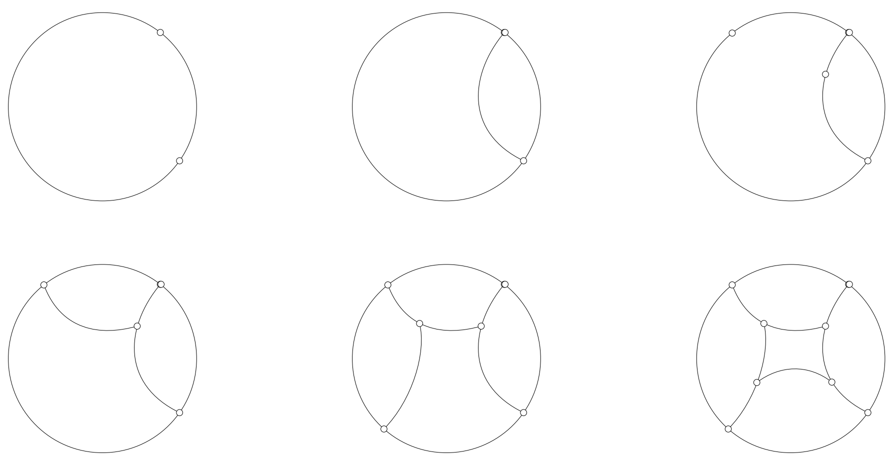

Figure 12: Simple "Polyhedron"

In Figure 12 we will illustrate the first few steps to convert Figure 11 to a distorted version of Figure 10.

In the upper left corner of the figure, a single additional point was added to the edge of our simple polyhedron. Next, in the center of the top row, a new edge was added that connects the two vertices. On the right side of the top row, two new vertices have been added: one on the newly drawn edge, and one on the original edge.

In the lower left of Figure 12, those new vertices are connected with a new edge. In the center of the lower line, two new vertices and a new edge were added. Finally, in the lower right, the conversion to Figure 10 is completed with the addition of two new vertices and a new edge.

Since every step was one of the two modifications guaranteed to leave the expression $F - E + V$ unchanged, since its initial value was 2, it will remain 2 though the complete transformation to the representation of a flattened cube, or to a flattened version of any of the other polyhedra discussed in this article. This proof seems a bit sloppy, but the main idea is there, and it can be modified to a rigorous proof of Euler's theorem.

Similar ideas could be used to prove particular cases. For example, if you knew that Euler's theorem held for the icosahedron, it's easy to show that it continues to hold for the truncated icosahedron in Section 7. Each time you cut off a corner of the original icosahedron, you cut off the old vertex and you introduce one new face, five new edges and five new vertices. The net change each time to $F - E + V$ is zero.

In Appendix A there are a few other examples of polyhedra that you can use as exercises in counting, and to verify that the formula $F - E + V$ holds in a few additional cases.

Finally, in Appendix B contains a few triangles, squares, pentagons, and hexagons. Copy this sheet on stiff paper, cut out the polygons and attach them together by the tabs to construct many of the examples here. Better still, check out this web site:

`http://www.korthalsaltes.com/`

9 The Game of Criss-Cross

This game is another good way to introduce Euler's Theorem. The idea (I think) is due to Sam Vandervelde, and his writeup for the game can be found here:

`http://www.mathteacherscircle.org/materials/crisscross.pdf`

The idea is this: begin with a triangle formed with three dots, and place some number of other dots inside that triangle such that no three of the dots lie on a straight line. Players take turns choosing a pair of dots (including the dots that formed the original triangle) and connecting them with a straight line segment such that each new segment does not cross any existing segment. The game ends when a player is unable to make a move, and that player loses. The question is, given the initial configuration of dots inside the triangle, which player wins?

A little experimentation will show that it doesn't matter how the dots inside the original triangle are arranged; the only thing that matters is the original number of dots. If there are an even number of dots, the first player wins; otherwise, the second player does. If there are n dots inside the initial triangle (making $n + 3$ total dots), then at the end of the game, there will be $3n + 3$ segments drawn, which alternates even and odd as n increases.

At the end of the game, there will be only triangluar faces, since if any polygon with more sides remains, a diagonal can be added to it. (We should probably call the exterior region a triangle, since although it is unbounded outside, the boundary it has consists of three segments.

There are a lot of ways to figure out the total number of segments, faces and edges is to imagine taking scissors and cutting up a completed game into a collection of triangles. After cutting, each original edge will be represented on two of the triangles, so if F represents the number of triangles, then $E = 3F/2$ represents the number of edges. If there were n dots inside, then there are $V = n + 3$ vertices in the completed graph. Since $F - E + V = 2$, we have:

$$F - (3F/2) + (n + 3) = 2.$$

Solving for F yields: $F = 2n + 2$, and this means that $E = 3n + 3$, as stated above.

This is a very short description of the game and its solution. See Sam Vandervelde's article for suggestions of how to use it in a classroom.

Following is a proof that for a game of Criss-Cross with n points inside the triangle will end with $3n+3 = 3(n + 1)$ edges, independent of knowing Euler's theorem. It begins with with same observation that if you were to cut a completed game into triangles with scissors and if you count the sheet of paper with a triangular hole as another triangle, then if F is the number of triangular faces and E is the number of edges, then:

$$3F = 2E,$$

since each triangle will have three edges, and each edge will be counted twice; once each on the two triangles on both sides of the edge.

What we are going to do is remove vertices inside the outer triangle one by one, and keep track of how the number of regions (which we will call F, that will include both triangles and more complex regions), edges (which we will call E) and vertices (which we will call V).

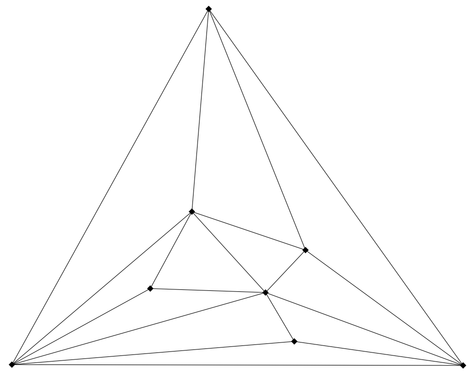

Figure 13: Typical end of game

The following analysis will apply to any diagram, but to follow along, consider the completed game as illustrated in Figure 13 from a game that began with five vertices inside the initial triangle. In this example, $V = 8 = 5 + 3$ (three initial vertices plus the five vertices added inside). We also have $F = 12$ (remember to count the outside triangle), and $E = 18$.

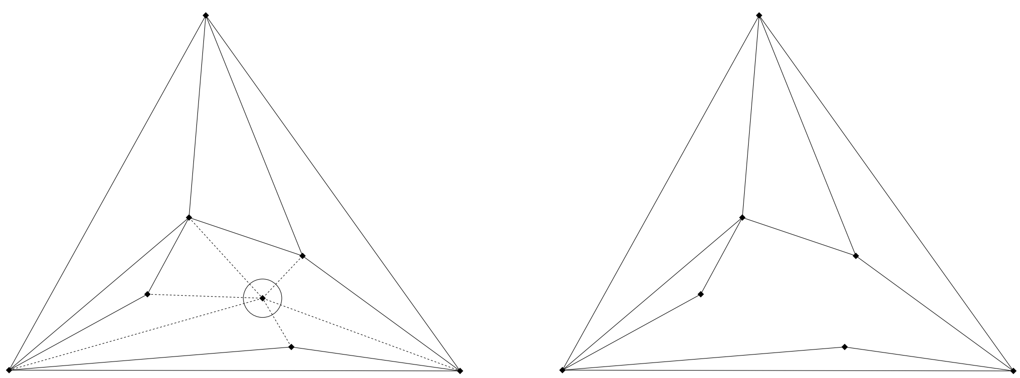

Figure 14: Removing a point

Now we will remove vertices one at a time, but as an example, suppose we first remove the circled vertex in the diagram on the left of Figure 14. When that vertex is removed we will also remove all the edges connected to it, which, on the left side of Figure 14, are the dotted lines. When that vertex and the connected lines are removed, the figure is changed to that on the right of Figure 14.

When an internal vertex is removed, the number of internal vertices in the resulting figure is reduced by 1, but what happens to the number of edges and regions? Obviously, this depends upon which internal vertex is removed. In the original example, two of the vertices have 3 connected edges, one has 5 and one has 6. We happen to have selected the vertex with 6 connected edges.

Using our concrete example, the number of edges removed will be 6, but what happens to the number of regions? (For our analysis, we consider “regions” to be any polygon, not necessarily a triangle, including the surrounding outer region.) If we consider the removed vertex as the center, the 6 edges coming out from it divide the larger region into 6 sub-regions. When we remove the center vertex, all 6 of those sub-regions disappear, but they are replaced by a single larger region, so six are subtracted and one is added.

There’s nothing special about 6; suppose that when the first vertex is removed, there are k_1 edges connected to it. Then after the removal, there will be k_1 fewer edges, and $k_1 - 1$ fewer regions (k_1 sub-regions are removed, and 1 larger region appears).

So let’s let V_1, E_1 and F_1 represent the number of vertices, edges and regions remaining after a single vertex with k_1 connected edges is removed. We have:

$$\begin{aligned} V_1 &= V - 1 \\ E_1 &= E - k_1 \\ F_1 &= F - k_1 + 1 \end{aligned}$$

But there is nothing special about removing the first vertex. When we remove the second, suppose that there are k_2 edges connected to it. Again, there will be one fewer vertex, k_2 fewer edges, and $k_2 - 1$ fewer regions, yielding:

$$\begin{aligned} V_2 &= V - 2 \\ E_2 &= E - k_1 - k_2 = E - (k_1 + k_2) \\ F_2 &= F - k_1 - k_2 + 2 = F - (k_1 + k_2) + 2, \end{aligned}$$

where V_2, E_2 and F_2 are the number of vertices, edges and faces remaining after the second removal. The same process can be repeated n times (since there are n internal vertices) and we will have the following sequence of formulas:

$V_1 = V - 1$	$E_1 = E - k_1$	$F_1 = F - k_1 + 1$
$V_2 = V - 2$	$E_2 = E - (k_1 + k_2)$	$F_1 = F - (k_1 + k_2) + 1$
$V_3 = V - 3$	$E_3 = E - (k_1 + k_2 + k_3)$	$F_1 = F - (k_1 + k_2 + k_3) + 1$
$\cdots$	$\cdots$	$\cdots$
$V_n = V - n$	$E_n = E - (k_1 + \cdots + k_n)$	$F_n = F - (k_1 + \cdots + k_n) + n$

Now, the nice thing about the final formula is that the quantity $k_1 + k_2 + \cdots + k_n$ represents the total number of internal edges since once all the internal vertices are removed, all the internal edges must also have been removed, leaving only the three edges connecting the points of the original triangle, or:

$$\begin{aligned} 3 &= E - (k_1 + k_2 + \cdots + k_n) \\ E - 3 &= (k_1 + k_2 + \cdots + k_n) \end{aligned}$$

Similarly, after all the internal vertices have been removed, only two regions remain: the inside and outside of the original triangle. Mathematically:

$$2 = F - (k_1 + k_2 + \cdots + k_n) + n.$$

Since $E - 3$ is the same as $(k_1 + k_2 + \cdots + k_n)$ we can write the equation above as:

$$2 = F - (E - 3) + n. \tag{2}$$

Recalling that $3F = 2E$, which we obtained by cutting the original figure into triangles and counting the resulting edges (each triangle has three, but each original edge is counted twice in the cut-up version), we'd like to combine this fact with Equation 2:

$$\begin{aligned} 2 &= F - (E-3) + n \\ 6 &= 3F - 3(E-3) + 3n \\ 6 &= 2E - 3E + 9 + 3n \\ E &= 3 + 3n, \end{aligned}$$

which is what we were trying to prove. Thus if the number of dots, n, is even, the number of edges will be odd and vice-versa. So if the number of dots is odd, the first person wins, and if the number of dots is even, the second person does.

10 Flattened polyhedra

This section is mostly a collection of drawings of some polyhedra that are flattened as described in the previous section. The first two examples are of a dodecahedron and an icosahedron and are illustrated in Figure 15. Note that the dodecahedron has 11 closed faces, each with 5 sides, and the "outside" is the twelfth face, which also has 5 sides. In the same way, the icosahedron has 19 (instead of 20) closed faces, all triangles, and the outside represents the twentieth face.

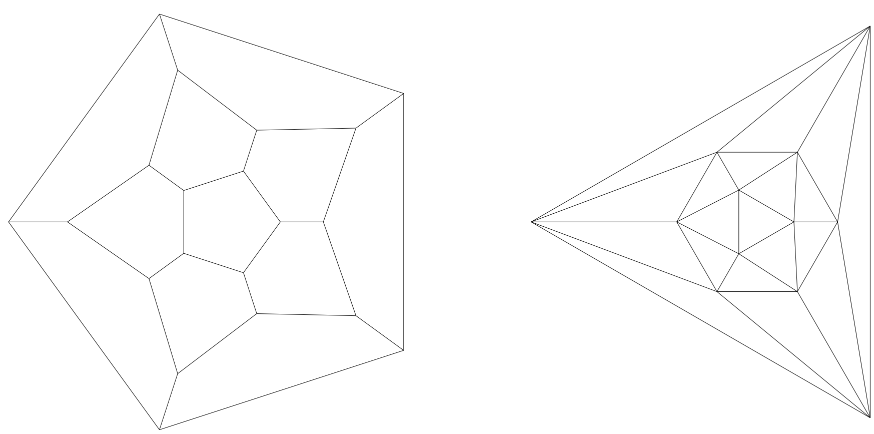

Figure 15: Flattened Dodecahedron and Icosahedron

A flattened version of the "soccer ball" from Section 7 appears on the left in Figure 16.

We have not examined this possibility yet, but another approach to flattening a polyhedron would be to cut a tiny hole that includes a vertex, and then to do the stretching. If you did this, the point that you removed would behave like a "point at infinity" and the lines connecting to it would stretch out forever from the main part of the figure. As an illustration of this, see the drawing on the right in Figure 16. In that figure, notice that there are still 20 "triangles", where 5 of them, on the outside, are infinite. All the arrows at the ends of the lines will meet at the removed point at infinity. Figures such as this also satisfy Euler's theorem, at least if we count the point at infinity as another point. Otherwise it will appear to be off by 1.

Finally, Figure 17 illustrates the invariance of Euler's theorem under duality, as described earlier in Section 4.1. The figure is basically a combination of the drawing on the left side of Figure 15 and the drawing on the right in Figure 16. One has points indicated by small open circles and the other by filled squares. There is a circle point in the center of every square polygon and vice-versa, assuming, of course, that there is circle at the "point at infinity". One figure is composed of solid line segments and the other of dashed segments, and every solid segment intersects exactly one dashed segment and vice-versa. Thus

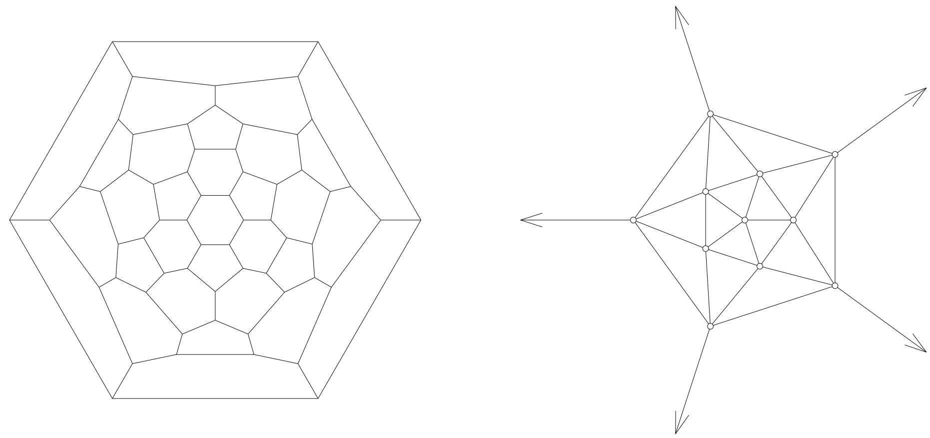

Figure 16: A Flattened "Soccer Ball" and Icosahedron with a Point at Infinity

the number of edges in both figures is the same. Similarly, every point in one figure corresponds to a face in the other and vice versa.

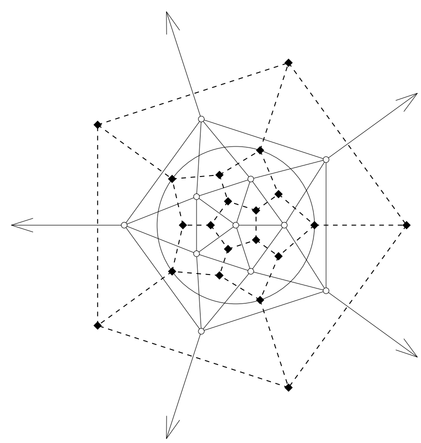

Figure 17: Duality of the Dodecahedron and Icosahedron

11 Another Proof of Euler's Theorem

The following proof is due to Martin Isaacs, and is much more beautiful than the one in the previous section. It is also interesting in that it actually proves a more general version of Euler's theorem and the more general form makes the proof easier. It is based on the idea of an invariant that is preserved as you remove edges from a figure. Isaacs's proof works on more general figures, which he called "scribbles". A scribble is any drawing of (possibly curved) lines and points such that:

- Every line segment (edge) has a vertex at each end. Note: a line segment must have endpoints: a circle with no points on it is not a valid part of a scribble, since it has no endpoints. There is no problem if a segment has the same endpoint at both ends, so a circle with a single vertex on it is a valid piece of a scribble.

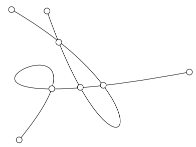

Figure 18: A Simple Scribble

- Whenever two or more segments cross, there is a vertex at the crossing that separates all the crossing edges into pieces.

Figure 18 illustrates a scribble that contains 8 vertices, 10 edges, and 4 faces, where the entire exterior is considered to be a single face. In this case, $V = 8$, $E = 10$ and $F = 4$, so we have the usual: $F - E + V = 2$, as it should.

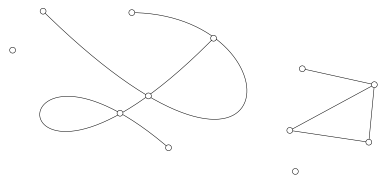

Figure 19: A More Complex Scribble

Isaacs, however, added the possibility that a scribble could consist of multiple "components", as illustrated in Figure 19. In that figure, there are four components, two of which are single vertices. A "component" is a set of vertices connected by edges. Two components are different if there is no connection by edges of the vertices of one with the vertices of the other. In the example in Figure 19, we have: $V = 12$, $E = 11$, $F = 4$, and $C = 4$, where "C" is the number of components.

Components are simply disconnected pieces of a scribble. One component could completely surround another, so a circle with a point on it plus a single point inside the circle would be a scribble that consists of two components (and two vertices, two faces, and one edge).

If you fool around with some examples (and at this point, it is a very good idea to do so), it appears to be the case that:

$$V - E + F - C = 1,$$

which is identical to our original version of Euler's formula in the case where $C = 1$. It has the additional advantage of working for the empty scribble, since in that case $C = V = E = 0$, but $F = 1$.

One other scribble that will turn out to be very interesting is the case where the scribble consists of nothing except for, say, n vertices: $V = n$. In that case, $F = 1$, $E = 0$, $C = V = n$, and we have: $V - E + F - C = n - 0 + 1 - n = 1$, as it should.

Isaac's proof works as follows: If we begin with a scribble with any degree of complexity (well, assuming there are at most a finite number of vertices, edges and faces), then we can show that if any single edge is removed from that scribble, the quantity $Q = V - E + F - C$ is invariant; in other words, the value of Q before and after the removal of the edge remains the same. So for any scribble that we begin with, although originally we don't know the value of Q, we show that successively simpler scribbles have the same value of Q, and when we eventually remove all the edges, we are left with a scribble that consists only of vertices, and we know that all such vertex-only scribbles satisfy $V - E + F - C = 1$.

When we remove a single edge from a scribble, there are two cases to consider. First, the removal of the edge might separate a single component into two. This will be the case if there is no other edge connecting the components. If the values for the original scribble are V, E, F and C, and the values for the new scribble with the edge removed are V', E', F' and C', it is easy to see that: $V' = V$, $E' = E - 1$, $F' = F$ and $C' = C + 1$. There are no vertices added or removed, one edge is removed, the number of faces is the same, since the removed edge simply "separated" the outside from itself, and there is one more component, since removing the edge split its original component into two. Thus:

$$Q' = V' - E' + F' - C' = V - (E - 1) + F - (C + 1) = V - E + F - C = Q,$$

so Q is invariant on removal of such an edge.

The other possibility is that the removal of the edge does not split a component into two pieces, since there is at least one other connection between the parts. In this case, $V' = V$ and $E' = E - 1$ for the same reasons as before. Now $C' = C$ since there is no new component created, and $F' = F - 1$, since the edge that was removed had to separate two different faces. If both sides of it were the "outside", then its removal would split the component. This time, we have:

$$Q' = V' - E' + F' - C' = V - (E - 1) + (F - 1) - C = V - E + F - C = Q,$$

so Q is again invariant on removal of this other type of edge and the proof is complete.

12 Application: Geodesic Domes

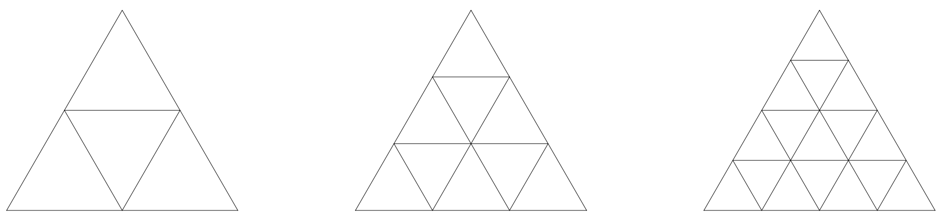

Figure 20: Uniform Triangle Subdivision

Geodesic domes were invented or at least popularized by Buckminster Fuller. They are composed only of triangles, so they are rigid. The usual dome is constructed by taking an icosahedron (see Figure 5), dividing each triangular face into smaller triangles, and then projecting the inner vertices from the center of the icosahedron to the sphere in which the original icosahedron could be inscribed. The resulting figure is then cut in half, or approximately in half, in case the triangles are divided into an odd number of subtriangles, and the result is a geodesic dome. Figure 20 shows how each triangle would be subdivided into $2^2 = 4$, $3^2 = 9$, or $4^2 = 16$ sub-triangles.

Figure 21: 2V and 3V Domes

Figure 21 shows the domes that result from subdividing each triangle into $2^2 = 4$ or $3^2 = 9$ sub-triangles. These are called, respectively, a 2V and a 3V dome.

If you look at every dome thus formed, it is obvious that there will be precisely 6 vertices where 5 edges come together (or vertices of degree 5) since at every subdivided vertex there will be 6 edges coming together and the original icosahedron had 12 vertices of degree 5. Since only half are used in a dome, there are $12/2 = 6$ vertices of degree 5.

This is equivalent to saying that every sphere approximation, before cutting it in half, contains 12 vertices of order 5.

What is perhaps somewhat amazing is that it is possible to form an infinte number of sphere-like figures from triangles such that every vertex has degree 5 or 6, and not just by using the standard geodesic dome design. Although the number of vertices of degree 6 in such sphere-like objects can be almost anything, there are always exactly 12 vertices of degree 5. Using Euler's theorem this is fairly easy to prove.

Let V_5 and V_6 represent the number of vertices in such a sphere-like object of degree 5 and 6, respectively. If we count all the outgoing edges, we obtain $5V_5 + 6V_6$, but this counts every edge twice, so the total number of edges is half that: $E = (5V_5 + 6V_6)/2$. The number of triangles adjacent to the edges is the same: $5V_5 + 6V_6$, but this triple-counts the triangles, since each will be counted adjacent to each of its vertices. Thus $F = (5V_5 + 6V_6)/3$. The total number of vertices, of course, is just $V = V_5 + V_6$.

Euler's theorem tells us that:

$$2 = V - E + F = V_5 + V_6 - \frac{5V_5 + 6V_6}{2} + \frac{5V_5 + 6V_6}{3}.$$

A bit of algebra yields:

$$\begin{aligned}
2 &= V_5 + V_6 - \frac{15V_5 + 18V_6}{6} + \frac{10V_5 + 12V_6}{6} \\
2 &= V_5 + V_6 - \frac{5V_5}{6} - \frac{6V_6}{6} \\
2 &= V_5/6 \\
12 &= V_5.
\end{aligned}$$

If you look at the dual of this result, where instead of a surface made of triangles we connect the centers of all the triangles and use the centers as the new vertices for a sphere-like object made only of polygons with 5 or 6 sides (like a soccer ball; see Figure 8), then there will be exactly 12 pentagons and an unknown number of hexagons (in fact, any number of them other than 1). It is sort of fun to try to draw figures like this: every vertex has three lines coming out of it, there are a prescribed number of hexagons: $(0, 2, 3, 4, \ldots)$, and there are exactly 12 pentagons. (Remember to count the outer pentagon or hexagon.) Figure 22 illustrates examples with 2, 3 and 5 hexagons. The soccer ball has exactly 20 hexagons and the dodecahedron has 0.

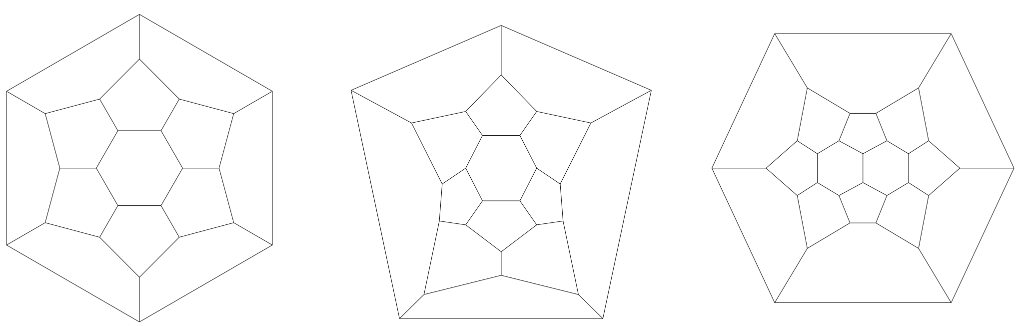

Figure 22: 2, 3 and 5 Hexagons Plus 12 Pentagons

It is a fairly easy exercise to show that in a figure of this sort with n hexagons and 12 pentagons there must be exactly $20 + 2n$ vertices. This is useful to know if you are trying to draw one.

13 Application: The Six-Color Theorem

A very famous theorem in mathematics states that for any map that can be drawn on a plane (or on a sphere), only four colors are required to paint the regions (which represent, say, countries) in such a way that guarantees that countries sharing a boundary (not just a point) are of different colors. At the time of this writing, the only proof of this theorem is based on the output of a computer program that checked thousands of cases to which the general problem had been reduced by mathematicians. So far there is not an easily-understood proof.

It is much easier to prove that five colors are sufficient, but what we will show here is that, as a relatively trivial result of Euler's theorem, that six colors are sufficient.

Map coloring can be reduced to a graph as follows: Place a vertex in the interior of every country and draw an edge from that vertex to the vertices on the interiors of all countries that share an edge by passing that line through the shared edge. This is a dual of the country map, and is clearly planar. If colors can be assigned to each vertex such that no two vertices connected by an edge have the same color then we have a valid map coloring. What we will show here is that every planar map has a valid coloring using six or fewer colors.

The proof is not too difficult, and we will begin with an outline, followed by the details.

The proof is based on induction on the size of the graph. If the graph has 6 or fewer vertices, it is obviously true. Assume that we know the theorem is true for any graph of size less than n, and for any graph of size $n > 6$ we will use Euler's theorem to show that the graph must have at least one vertex of degree 5 or less. If we remove that vertex, and all the edges coming from it, we will have a graph of size $n - 1$ which we know is 6-colorable by the induction hypothesis. If we add the vertex and edges we just removed, the new vertex has only 5 or fewer neighbors, so there will be a color available for it that will not conflict with the other 5 or fewer.

So all we really need to show is that any planar graph of size 6 or larger has a vertex with 5 or fewer neighbors. We will do this by using Euler's theorem to show that for any planar connected graph with 3 or more vertices that $E \leq 3V - 6$. Then if we want to find the average degree of a vertex, we add all the degrees and divide by the total number of vertices. Each edge has two vertices at its endpoints, so the average degree D is given by:

$$D = \Big(\sum_{v \in V} deg(v)\Big)/V = (2E/V) \leq (2(3V - 6))/V = 6 - 6/V.$$

This number is strictly smaller than 6 so at least one vertex must have degree smaller than 6.

All we need to do, then, is to prove that for any planar connected graph with 3 or more vertices, that $E \leq 3V - 6$.

If the graph has no cycles, then $E = V - 1 < V$. Since $V \geq 3$ then $2V - 6 \geq 0$. Add the inequalities $E < V$ and $0 \leq 2V - 6$ to obtain $E < 3V - 6$, so we are done in this case.

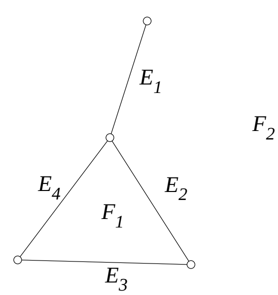

Figure 23: Face and Edge Example

If the graph does have cycles, consider the set of all possible edge-face pairs, where the edge and the face touch. In Figure 23 there are four edges and two faces (F_2 is the "outside" face). In this example, all the edges are adjacent to F_2, but only E_2, E_3 and E_4 are adjacent to face F_1. Thus the complete set of edge-face pairs for this example would consist of the following 7 pairs:

$$S = \{(E_1, F_2), (E_2, F_2), (E_3, F_2), (E_4, F_2), (E_2, F_1), (E_3, F_1), (E_4, F_1)\}.$$

Note that each face must be adjacent to at least 3 edges. If there were only two, that would mean that the same two vertices were connected by multiple edges, and we only connect faces that share a boundary once. Thus the set S must consist of at least $3F$ elements: $|S| \geq 3F$, where $|S|$ denotes the number of elements in S.

Each edge touches at most two faces, and in the example in Figure 23 we see that E_1 touches only one face, so $|S| \leq 2E$. Combining this with the result in the previous paragraph: $3F \leq 2E$.

But if we multiply Euler's formula by 3 and substitute for the resulting $3F$, we obtain:

$$\begin{aligned} 2 &= V - E + F \\ 6 &= 3V - 3E + 3F \\ 6 &\leq 3V - 3E + 2E \\ 6 &\leq 3V - E \\ E &\leq 3V - 6, \end{aligned}$$

which is the result we needed to prove the six-color theorem.

A Additional Figures

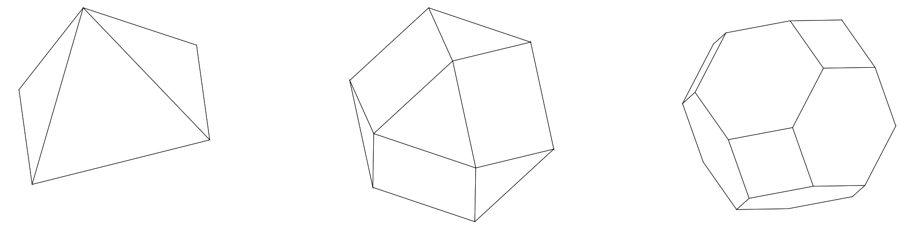

Figure 24: Egyptian Pyramid, Cuboctahedron, Truncated Octahedron

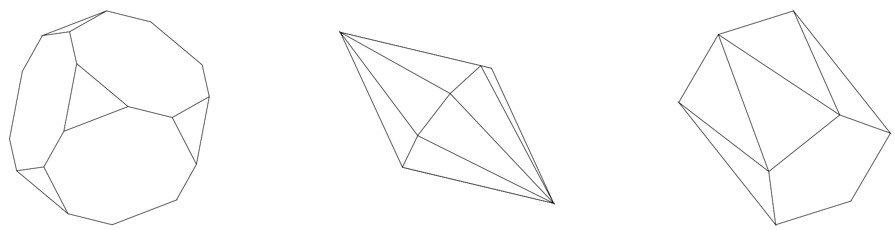

Figure 25: Truncated Cube, 8-Sided Bipyramid, Pentagonal Antiprism

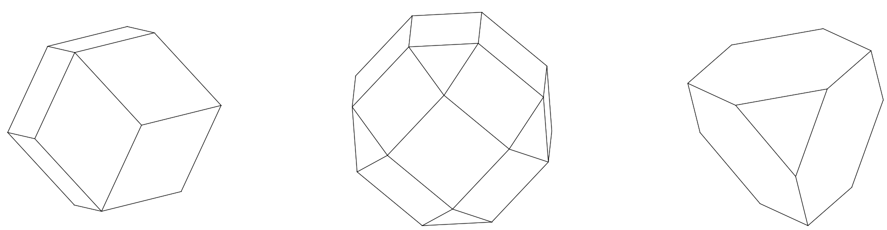

Figure 26: Rhombic Dodecahedron, Rhombicuboctahedron, Truncated Tetrahedron

B Polygon Cutouts