Counting Things

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Abstract

We present here various strategies for counting things. Usually, the "things" are patterns, or arrangements. For example, "How many ways can you choose 3 balls from a collection of 6 balls?"

1 General Strategies

Following this initial section on general strategies every section will begin with a short set of problems whose solution illustrates a different technique for counting things. The solutions appear in another section later in the document. The best way to read this document is to look at the problems and try to solve them yourself before going on to read the solutions. At the end (see Section 10) is a set of miscellaneous problems without solutions.

Especially at first, when you encounter a problem, you will have no idea how to proceed. Here are some general methods. Keep in mind that many problems require a combination of these methods.

- Solve a few tiny problems of the same sort and see if you can find a pattern. For example, if you don't know anything about combinations and you are asked to find the number of ways to choose six numbers from a lottery ticket with 51 numbers on it, try solving the problem for "lotteries" where you choose $1, 2,$ or 3 numbers from collections of $1, 2, 3, 4,$ or 5 possibilities. See Section 2.
- Can you break the problem into parts? In other words, if the things you are counting fall into distinct classes with no overlap, you can count the things in each class and add the results together. For example, suppose you are counting paths through a city, but you notice that every path has to go through one of two intersections. If you count the paths through each intersection somehow, you can add those numbers together to obtain the total count. See Section 3.
- If the items you are counting have independent parts, you can count the number of each kind of part and multiply the results. For example, how many license plates are there that begin with a letter of the alphabet and are followed by a 6-digit number? Well, there are 26 ways to choose the letter, and there are a million 6-digit numbers, so there are $26 \times 1000000 = 26000000$ possible license plates. See Section 3.
- There are standard formulas for counting combinations and permutations that will be discussed later in this paper. See Section 4.
- Sometimes it is easier to count the things you do *not* want. For example, suppose the problem is to count all the 5-digit numbers that contain at least one 7. This is a mess since the number can contain $1, 2, 3, 4,$ or 5 7s, but you know there are 100000 total numbers, and it is easy to count the number that have no 7s in them (there are $9^5 = 59049$ of these, so there are $100000 - 59049 = 40951$ numbers containing at least one 7. See section 5.
- Sometimes problems that seem totally different are actually equivalent. For example, problems 36 and 38 are the same, since counting the routes through the city is equivalent to dividing the movements of one block south into each of the possible north-south streets. In other words, problem 36 is a special case of problem 38 where $k = 6$ and $n = 9$.

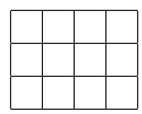
2 Organization (Discussion: see Section 6)

1. How many ways can you choose 1 thing from a set of 2? of 3? of 4? of n?

2. Make a list of all the ways to choose 2 things from a set of 2. From a set of 3. From a set of 4. From a set of n. How can you be certain that your list is complete? Is the list arranged in some logical order to make certain you have not left our any combinations?

3. Make a list of all orderings of 4 items. "Ordering" is simply the arrangement. For example, here are a few of the orderings of 5 items: 12345, 13245, 54321, 41235, et cetera. (Remember to find a logical listing, and it might be a good idea to begin with listing the orderings of 1, 2, and 3 items.)

4. Make a list of all of the 35 shortest paths from the upper left corner to the lower right corner of the following grid:



   One reasonable way to begin is to notice that every step is either down (D) or to the right (R). Thus one possible path will be: $DDDRRRR$.

3 Adding and Multiplying (Discussion: see Section 7)

5. A, B, and C are cities. If there are 4 roads from ($A \Rightarrow B$) and 3 from ($B \Rightarrow C$), how many routes are there from ($A \Rightarrow C$)? (Assume that all roads are one-way, in the direction of the arrows.)

6. If, in addition to the roads listed in the problem above, there are 6 roads from ($C \Rightarrow D$), how many ways can you travel from ($A \Rightarrow D$)?

7. As in the problem above, but 4 from ($A \Rightarrow B$), 3 from ($B \Rightarrow C$), 5 from ($A \Rightarrow D$), and 5 from ($D \Rightarrow C$). How many ways can you travel from ($A \Rightarrow C$)?

8. In how many ways can you choose a captain and co-captain of a football team with 11 members, assuming that the captain and co-captain are different people, and that the choice (captain=Tom and co-captain=Fred) is different from the choice (captain=Fred and co-captain=Tom)?

9. In how many ways are there to choose a president, a vice-president, and treasurer from a club of 15, assuming all three are different people?

10. In how many ways to put a white and black rook on a chessboard so that neither can attack the other? (Rooks can only attack along rows and columns—not along the diagonals.)

11. In how many ways to place a white and black king on a chessboard so that neither attacks the other? (A king attacks only those squares adjacent to it, so a king away from the edge of the board attacks the 8 adjacent squares.)

12. If you have an alphabet of 26 letters, how many 3-letter words can you make? What if the three letters all have to be different? How many 5 letter words can you make, if you can repeat letters, but cannot have 2 in a row that are the same?

4 Permutations and Combinations (Discussion: see Section 8)

13. How many four digit numbers are there that contain the digits $1, 2, 3$, and 4 in some order?
14. How many ways can you put 8 mutually non-attacking rooks on a standard 8×8 chessboard?
15. How many rearrangements can be made of the letters in the following words: VECTOR, TRUST, CARAVAN, CLOSENESS, MATHEMATICAL? (For example, for "VECTOR", some possibilities include: VECTRO, OTCEVR, and ROTVEC.)
16. How many ways are there to choose a team of 3 students from a group of 30?
17. How many ways can a group of 10 girls be divided into two basketball teams of 5 girls each?
18. One student has 6 books and another has 8. In how many ways can they exchange 3 books of the first student for 3 books of the second?

5 Subtracting (Discussion: see Section 9)

19. How many 6-letter "words" contain at least one letter "A" (if any sequence of letters counts as a word)?
20. How many whole numbers are there from 0 to 10000 that do not have any factors of 2 or 3?
21. How many whole numbers are there from 0 to 10000 that do not have any factors of 2, 3, or 5?

6 Organization: Discussion

The main idea here is to come up with a plan for listing the results in a logical way when you are counting small sets to be certain that you haven't left out anything. If, for example, you want to list all the ways to choose 3 things from a set of 6 and you just start listing the ones that pop into your head, it is very difficult to know for sure that you have gotten all of them. There are 20 in total, and if you see a list of 19, it is almost impossible to see which one is missing.

One good method is to arrange them in "alphabetical order". For example, suppose we want to list all the subsets with three elements of the set: $\{a, b, c, d, e\}$. Here is the "alphabetical" listing:

$$abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde.$$

Make sure you understand how to make listings like this.

If you have never done this before, work out all the examples in problems 1 through 4. For problem 2 there are 1, 3, 6, and $n(n-1)/2$ solutions. Problem 3 there are 24 orderings.

7 Adding and Multiplying: Discussion

In problem 5, every route must pass through city B, and the choice you make to get from A to B is independent of the choice you make to get from B to C. Thus, for *every* one of the 4 roads from A to B, there are 3 from B to C, so the grand total is $4 \times 3 = 12$.

In problem 6, you can use the same reasoning, but think of it as follows: You already calculated that there are 12 routes from A to C, and for every one of those routes, there are 6 routes from C to D. Thus the answer is $12 \times 6 = 72$ total different ways to move from A to D.

To solve problem 7, we notice that to get from A to C, we must pass through B or through D. Thus we can count (as in problems 5 and 6) the number of routes through B ($4 \times 3 = 12$) and through D ($5 \times 5 = 25$) and add those together to obtain $12 + 25 = 37$ total routes.

It may be helpful to think of the combinations of routes in problems 5, 6 and 7 in terms of the logical operators "**AND**" and "**OR**". In problem 5 our route needs to move from A to B **AND** from B to C. In problem 7 the route passes through B **OR** D. Usually, **AND** corresponds to multiplication in problems like this, and **OR** to addition.

Thus, here is a description of the routes for problems 5, 6 and 7 expressed as mathematical equations. On the left is a logical description of the possible routes, and on the right is a translation to a formula where **AND** and **OR** have been replaced by $\times$ and $+$, respectively.

$$\begin{aligned}
(A \Rightarrow B) \textbf{ AND } (B \Rightarrow C) &= 4 \times 3 = 12 \\
(A \Rightarrow B) \textbf{ AND } (B \Rightarrow C) \textbf{ AND } (C \Rightarrow D) &= 4 \times 3 \times 6 = 72 \\
((A \Rightarrow B) \textbf{ AND } (B \Rightarrow C)) \textbf{ OR } & \\
((A \Rightarrow D) \textbf{ AND } (D \Rightarrow C)) &= (4 \times 3) + (5 \times 5) = 37.
\end{aligned}$$

Problems 8 and 9 are actually similar to the first problems. In problem 8, for example, we begin without having chosen anyone, and we will consider "routes" that get us to the condition of having chosen a captain (there are 11 ways to do this), and then we need to move from there to the condition of having also chosen a co-captain. It's a little different in that each different choice of the captain leaves a different set of 10 choices for co-captain, but in every case, there are 10 different paths to choose. Thus there are $11 \times 10 = 110$ total choices of captain and co-captain.

In problem 9, similar reasoning gives $15 \times 14 \times 13 = 2730$ total ways to choose all three club officers.

Problem 10 seems different, but it is not. You can place the white rook on any of the 64 squares, but as soon as you have placed it, you cannot place the black rook on any of the squares in the same row or column as the white rook. In fact, 15 squares are always eliminated, so there are $64 - 15 = 49$ possible placements of the black rook for every placement of the white rook. Thus the answer is $64 \times 49 = 3136$ possible arrangements.

Problem 11 is like problem 7. When you place the white king on the board, it eliminates a number of squares available to the black king, but the number of squares eliminated depends on where the white king is placed. If it is placed away from the edge, it eliminates 8 neighboring squares, so (including the square where it is placed), the other king can only be put on $64 - 9 = 55$ squares. If the white king is in a corner, it eliminates 4 possible squares, including the one in the corner, so the black king can only be placed on $64 - 4 = 60$ squares. For a white king on the edge but not in a corner, there are 6 squares eliminated, so there remain $64 - 6 = 58$ squares.

There are 36 interior squares, 24 edge squares that are not in a corner, and 4 corner squares. Using the logic of problem 7, there are $36 \times 55 + 24 \times 58 + 4 \times 60 = 3612$ arrangements of non-attacking kings.

Finally, problem 12 is like the previous ones. There are $26 \times 26 \times 26$ three-letter words, $26 \times 25 \times 24$ three-letter words where all the letters are different, and $26 \times 25 \times 25 \times 25 \times 25$ five-letter words where no two letters in a row can be the same. (The first letter can be chosen freely, but after that, there is always a single letter that cannot be used, leaving 25 possibilities for each successive choice.)

8 Permutations and Combinations: Discussion

To solve problem 13, notice that the first number can be any of 4, and after it is chosen, there remain 3 possibilities. After the second is chosen, there remain 2, et cetera. Thus, there are $4 \times 3 \times 2 \times 1 = 24$ rearrangements.

Problem 14 seems different from the previous problem, but in fact it is not. You place a rook on the first row in any of the 8 different squares, but once you pick a square, you have eliminated that square's entire column, so there are only 7 available squares in the second row. Placing this second rook eliminates yet another column, and so on. The total number of non-attacking rook arrangements is thus $8 \times 7 \times 6 \times \cdots \times 2 \times 1 = 40320$.

There is a standard mathematical notation for the product of all the integers from 1 to n, and that is $n!$ which is read, "n factorial". In the rook example above, there are $8!$ arrangements. $4! = 4 \times 3 \times 2 \times 1 = 24$, et cetera. It may not seem logical now, but there are some *very* good reasons to define $0!$ ("zero factorial") to be 1.

In problem 15, things are straight-forward if all the letters in the word are different. For example, the word VECTOR has 6 different letters, so the first letter of a rearrangement can be chosen in any of 6 ways, leaving 5 choices for the second letter, 4 for the third, and so on. Thus, there are $6 \times 5 \times \cdots \times 1 = 6! = 720$ rearrangements of the letters of the word VECTOR.

Problems arise, however, if there are duplicates. Rather than consider one of the example problems, let's look at a simple case: how many rearrangements are there of the word TEE? One approach is to imagine first that the two Es are different: TEe, for example. There are $3 \times 2 \times 1 = 6$ rearrangements:

TEe ⇔ TeE, ETe ⇔ eTE, EeT ⇔ eET

But notice that for every position of the T, the E and e can be arranged in two ways (the two versions on opposite sides of the "⇔" symbol above). That means that in the 6 total rearrangements with the two different types of E, every example is counted twice, so to get the correct answer, we need to divide by 2, yielding 3 different rearrangements:

TEE, ETE, and EET.

If there were three different copies of E in the original word, each rearrangement with three different types of E would yield 6 different versions corresponding to the 6 rearrangements of three items.

Similarly, if a word contains three Es and four Ts, we need to divide by $3!4!$ to get the true result. Consequently, here are the solutions to all the examples in problem 15:

$$\begin{aligned}
\text{VECTOR} &: 6! = 720 \\
\text{TRUST} &: 5!/2! = 360 \\
\text{CARAVAN} &: 7!/3! = 840 \\
\text{CLOSENESS} &: 9!/(3!2!) = 30240 \\
\text{MATHEMATICAL} &: 12!/(3!2!2!) = 19958400
\end{aligned}$$

To solve problem 16, we can begin by choosing three students in order. The first can be chosen in any of 30 ways; the second in any of 29, and the third in 28 ways. Thus at first glance, there are $30 \times 29 \times 28$ possible choices, but notice that for any particular group of three students, this method includes all possible rearrangements.

If three of the students are named A, B, and C (pretty dull names), then the method above includes all of these choices: ABC, ACB, BAC, BCA, CAB, and CBA. This group has been counted 6 times, once for each of the possible rearrangements. But the same thing will occur with *every* set of three students. Every set is counted 6 times. Thus the number $30 \times 29 \times 28$ is 6 times larger than it should be, so the real answer is $(30 \times 29 \times 28)/6 = 4060$.

Problem 17 is almost the same sort of problem. Notice that once you have chosen the first team, the second team is completely determined. There are $10 \times 9 \times 8 \times 7 \times 6 = 30240$ ways to choose the girls for one team, but each set of 5 girls will be picked in many orderings—$5! = 120$ of them, to be exact. Thus the number of ways of choosing the first team is $30240/120 = 252$.

There are 252 ways to pick 5 girls on team A and to put the other 5 on team B. If you don't care that the teams are called A and B, you should probably divide the 252 by 2 for your answer. If you don't divide by two, then you are considering these two divisions to be different: Girls $1, 2, 3, 4$, and 5 on team A (with the others on team B), and Girls $1, 2, 3, 4$, and 5 on team B (with the others on team A).

Either answer (252 or $252/2 = 126$) is correct, depending on exactly what is meant by the answer. As stated, the question does not make it perfectly clear.

Finally, in problem 18, the exchange is completely determined when each student has chosen the books to exchange. The student with 6 can do this in $(6 \times 5 \times 4)/(3 \times 2 \times 1) = 20$ ways, and the student with 8 can do it in $(8 \times 7 \times 6)/(3 \times 2 \times 1) = 56$ ways. Since the choices are independent, and any choice of one student is compatible with any choice of the other, there are $20 \times 56 = 1120$ solutions.

Notice that in the last few problems we have done the same thing over and over—we have counted the number of ways to choose k things from a set of n things. The answer is always obtained by multiplying n by $n - 1$ by $n - 2$ and so on, until we have k terms, and then dividing the result by $k!$. This is done so often in combinatorics that there is a special symbol for this operation: $\binom{n}{k}$, which is read aloud as "n choose k". It is equal to:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Be sure to understand why this calculation works.

9 Subtraction: Discussion

In problem 19, it is easier to count the patterns that have no copies of the letter A. There are $26^6 = 308915776$ ways to produce 6-letter words with all the letters of the alphabet. There are $25^6 = 244140625$ ways to choose combinations that have no copies of the letter A. The difference: $308915776 - 244140625 = 64775151$, is the number of words with at least one A in them.

Problems 20 and 21 illustrate a different technique. Let's look at a slightly easier problem first: How many numbers are there between 1 and 10000 that have no factors of 2, of 3, of 5? Obviously, half the numbers have no factor of 2 (the odd numbers), so the answer is 5000. How many have no factors of 3? Well, there are 3333 multiples of 3, so $10000 - 3333 = 6667$ of the numbers have no multiples of 3. Similarly, there are 8000 numbers in the range with no multiples of 5.

But if we try to answer problem 20 by beginning with 10000 and subtracting the 5000 multiples of 2 and the 3333 multiples of 3, the 1667 is incorrect, since we have subtracted *twice* the numbers that are multiples of both 2 and 3. Thus to get the correct answer, we must add all of these in. Numbers that are multiples of 2 and 3 are multiples of 6, and between 1 and 10000 there are 1666 of these. Thus the correct answer is $1667 + 1666 = 3333$.

Problem 21 is even messier. If we subtract the multiples of 2, of 3, and of 5, we have to add back in the number of multiples of $2 \times 3 = 6$, of $2 \times 5 = 10$, and of $3 \times 5 = 15$. But this will subtract out three copies of the multiples of $2 \times 3 \times 5 = 30$, so these have to be added back in.

Figure 1 shows what is going on. We want to count the items inside the largest circle, but outside the inner three. If we simply subtract the inner three, the central region (containing multiples of 2, 3, and 5) is subtracted 3 times.

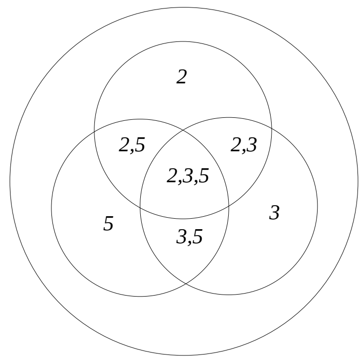


Figure 1: Venn Diagram

The regions labeled $2, 5$ or $2, 3$ or $3, 5$ are subtracted twice. So if we add in copies of the $2, 5$ and $2, 3$ and $3, 5$ regions, we add in the $2, 3, 5$ region three more times. That center region was added three times initially, and now is added back 3 times, so it needs to be subtracted once more. Here's the formula for the grand total, where we use the notation S_2 to mean the count of multiples of 2, $S_{2,3}$ to mean the count of numbers that are multiples of 2 and 3, et cetera. In the formula below, C is the final count, and S is the total of all the numbers, 10000 in our case:

$$C = S - S_2 - S_3 - S_5 + S_{2,3} + S_{2,5} + S_{3,5} - S_{2,3,5}.$$

10 Miscellaneous Problems

Warning: Some of the problems that follow are quite difficult. Difficult problems are marked with a $\star$. More difficult problems with $\star\star$, and so on.

Complete solutions to these problems appear at:

http://www.geometer.org/mathcircles/solnscount.pdf

22. How many diagonals are there in a convex n-gon?
23. There are 3 rooms in a dormitory, a single, a double, and a quad. How many ways are there to assign 7 people to the rooms?
24. How many 10-digit numbers have at least 2 equal digits?
25. How many ways can you put 2 queens on a chessboard so that they don't attack each other? (Queens attack both on the rows and on the diagonals of a chessboard.)
26. How many ways can you split 14 people into 7 pairs?
27. There are N boys and N girls in a dance class. How many ways are there to pair them all up?
28. Ten points are marked on the plane so that no three of them are in a straight line. How many different triangles can be formed using these 10 points as vertices?

29. A group of soldiers contains 3 officers, 6 sergeants, and 30 privates. How many ways can a team be formed consisting of 1 officer, 2 sergeants, and 20 privates?

30. Ten points are marked on a straight line and 11 on another line, parallel to the first. How many triangles can be formed from these points? How many quadrilaterals?

31. How many ways can you put 10 white and 10 black checkers on the black squares of a checkerboard?

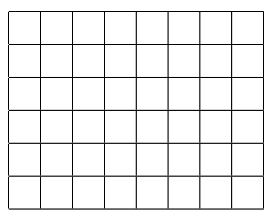
32. $\star$ How many 10-digit numbers have the sum of their digits equal to 1? The sum equal to 2? To 3? To 4?

33. To win the California lottery, you must choose 6 numbers correctly from a set of 51 numbers. How many ways are there to to make your 6 choices?

34. A person has 10 friends. Over several days he invites some of them to a dinner party in such a way that he never invites exactly the same group of people. How many days can he keep this up, assuming that one of the possibilities is to ask nobody to dinner?

35. There are 7 steps in a flight of stairs (not counting the top and bottom of the flight). When going down, you can jump over some steps if you like, perhaps even all 7. In how many different ways can you go down the stairs?

36. $\star$ The following illustration is a map of a city, and you would like to travel from the lower left to the upper right corner along the roads in the shortest possible distance. In how many ways can you do this?



37. In how many ways can 12 identical pennies be put in 5 purses? What if none of the purses can be empty?

38. In how many ways can you put k identical things into n boxes, where the boxes are numbered $1, \cdots, n$? What if you must put at least one thing in each box (so, of course, $k > n$)?

39. A bookbinder must bind 12 identical books using red, green, or blue covers. In how many ways can this be done?

40. $\star$ A train with M passengers must make N stops. How many ways are there for the passengers to get off the train at the stops? What if we only care about the number of passengers getting off at each stop?

41. How many ways are there to arrange 5 red, 5 green, and 5 blue balls in a row so that no two blue balls lie next to each other?

42. How many ways are there to represent 100000 as the product of 3 factors if we consider products that differ in the order of factors to be different?

43. There are 12 books on a shelf. In how many ways can you choose 5 of them so that no two of the chosen books are next to each other on the shelf?

44. In how many ways can a necklace be made using 5 identical red beads and 2 identical blue beads?

45. Given 6 vertices of a regular hexagon, in how many ways can you draw a path that hits all the vertices exactly once?

46. Within a table of m rows and n columns a box is marked at the intersection of the p^{th} row and q^{th} column. How many of the rectangles formed by the boxes of the table contain the marked box?

47. $\star$ A $10 \times 10 \times 10$ cube is formed of small unit cubes. A grasshopper sits in the center O of one of the corner cubes. At a given moment, it can jump to the center of any of the cubes which has a common face with the cube where it sits, as long as the jump increases the distance between point O and the current position of the grasshopper. How many ways are there for the grasshopper to reach the unit cube at the opposite corner?

48. Find the number of integers from 0 to 999999 that have no two equal neighboring digits in their decimal representation.

49. How many ways are there to divide a deck of 52 cards into two halves such that each half contains exactly 2 aces?

50. How many ways are there to place four black, four white, and four blue balls into six different boxes?

51. In Lotto, 6 numbers are chosen from the set $\{1, 2, \ldots, 49\}$. In how many ways can this be done such that the chosen subset has at least one pair of neighbors?

52. Given a set of $3n + 1$ objects, assume that n are indistinguishable, and the other $2n + 1$ are distinct. Show that we can choose n objects from this set in 2^{2n} ways.

53. In how many ways can you take an odd number of objects from a set of n objects?

54. n persons sit around a circular table. How many of the $n!$ arrangements are distinct, i.e., do not have the same neighboring relations?

55. $\star\star$ $2n$ points are chosen on a circle. In how many ways can you connect them all in pairs such that none of the segments overlap?

56. $\star\star$ In how many ways can you triangulate a convex n-gon using only the original vertices?

57. $\star\star$ If you have a set of n pairs of parentheses, how many ways can you arrange them "sensibly". For example, if you have 3 pairs, the following 5 arrangements are possible: ((())), (())(), ()(()), (()()), ()()().

58. $\star\star$ How many subsets of the set $\{1, 2, 3, \ldots, N\}$ contain no two successive numbers?

59. How many ways are there to put seven white and two black billiard balls into nine pockets? Some of the pockets may be empty and the pockets are considered distinguishable.

60. $\star\star$ How many ways are there to group 4 pieces of luggage? 5 pieces? (Here are the groupings of 3 pieces, A, B, and C: ABC, $A|BC$, $B|AC$, $C|AB$, $A|B|C$. The vertical bars represent divisions into groups.)

61. $\star$ Find the number of poker hands of each type. For the purposes of this problem, a poker hand consists of 5 cards chosen from a standard pack of 52 (no jokers). Also for the purposes of this problem, the ace can only be a high card. In other words, the card sequence $A\clubsuit$, $2\diamondsuit$, $3\diamondsuit$, $4\spadesuit$, $5\heartsuit$ is not a straight, since the ace is a high card only. The suit of a card is one of: $\clubsuit$, $\diamondsuit$, $\heartsuit$, or $\spadesuit$. The rank of a card is the number or letter: $2, 3, \ldots, 10, J, Q, K, A$.

    Here are the definitions of the hands followed by an example of each in parenthesis. The hands are listed here in order with the most powerful first. If a hand satisfies more than one of these, it is classified as the strongest class it satisfies. For example, the hand (9$\clubsuit$, 9$\diamondsuit$, 9$\spadesuit$, 6$\spadesuit$, 2$\heartsuit$) is certainly a pair, but it is also three of a kind.

Royal flush: 10 through A in the same suit. (10$\spadesuit$, $J\spadesuit$, $Q\spadesuit$, $K\spadesuit$, $A\spadesuit$)

Straight flush: 5 cards in sequence in the same suit. (4$\diamondsuit$, 5$\diamondsuit$, 6$\diamondsuit$, 7$\diamondsuit$, 8$\diamondsuit$)

Four of a kind: Four cards of the same rank. ($Q\spadesuit$, $Q\heartsuit$, $Q\diamondsuit$, $Q\clubsuit$, 7$\spadesuit$)

Full house: Three cards of one rank and two of another. (3$\spadesuit$, 3$\diamondsuit$, 3$\clubsuit$, 9$\heartsuit$, 9$\diamondsuit$)

Flush: Five cards in the same suit. (3$\clubsuit$, 4$\clubsuit$, 5$\clubsuit$, 6$\clubsuit$, 8$\clubsuit$)

Straight: Five cards in sequence. (6$\clubsuit$, 7$\clubsuit$, 8$\diamondsuit$, 9$\spadesuit$, 10$\clubsuit$)

Three of a kind: Three cards of the same rank. ($J\clubsuit$, $J\diamondsuit$, $J\spadesuit$, 7$\diamondsuit$, $K\heartsuit$)

Two pairs: Two pairs of cards. (5$\heartsuit$, 5$\spadesuit$, 8$\diamondsuit$, 8$\clubsuit$, $A\spadesuit$)

Pair: A single pair of cards. (3$\clubsuit$, 3$\diamondsuit$, 5$\spadesuit$, 9$\diamondsuit$, $Q\diamondsuit$)

Bust: A hand with none of the above. (2$\clubsuit$, 4$\spadesuit$, 6$\diamondsuit$, 8$\heartsuit$, 10$\clubsuit$)