Complex Numbers

Tom Davis
tomrdavis@earthlink.net
http://www.geometer.org/mathcircles
December 5, 2014

1 What is a Complex Number?

A complex number has the form $a + bi$, where a and b are real numbers and i is $\sqrt{-1}$. The number i (and real multiples of i) are called "imaginary numbers". In the complex number $a + bi$, a is called the "real part" and bi is called the "imaginary part."

Using the word "imaginary" is somewhat unfortunate, especially when used in conjunction with "real" since this carries the connotation that it's not an actual number. All numbers are human constructs and any one is as real as any other.

But every type of number is "unreal" from a certain point of view. The natural numbers, $\{0, 1, 2, 3, \ldots\}$, seem pretty "natural," but is the number:

$$10^{10^{100}}$$

(sometimes called a "google plex") "natural"? It is so huge (and there are plenty of natural numbers that are much larger) that it represents more items than there are particles in the physical universe.

2 Why Do We Need i?

The introduction of imaginary and complex numbers is no stranger than the introduction of the other types of numbers we use. Here is why various sets of numbers were invented.

- If we want to count things, we need the counting (or natural) numbers.
- If we want the operation $+$ and $-$ to work in every situation with the natural numbers, we need to add the negative numbers (thus creating the integers). Otherwise, we cannot compute things like $3 - 7$ or solve equations like $10 + x = 5$. Thus we need the integers.
- If we want division to make sense, we need the rational numbers. Otherwise, we can't calculate numbers like $5/7$ or solve equations like $7x = 5$. Therefore we need the rational numbers (fractions).
- If we want to talk about the exact circumference of a circle with radius 1 or if we want to talk about limits of sequences or have the ability to solve equations like $x^2 = 2$ we need the real numbers.

- If we want to be able to solve equations like $x^2 = -1$ we need imaginary numbers. To solve general quadratic equations (and much, much more) we need complex numbers.
- There are systems of even larger sets of numbers (like the quaternions) that arise, but in this article, we'll stop with the complex numbers.

Probably one of the most amazing things about complex numbers is that although they were introduced to solve equations like $x^2 = -1$ and quadratic equations like $x^2 + 4x + 5$ they are in fact capable of representing the solutions to any polynomial equations like $x^2 = i$ or $ax^7 + bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h = 0$, where $a, b, c, \ldots, h$ are themselves complex numbers. The addition of i and all combinations of i and real numbers provides a complete set of solutions for any polynomial equations.

If you haven't thought about it before, can you find $\sqrt{i}$ in the complex numbers? (The solution appears at the end of Section 7.)

3 Arithmetic in the Complex Numbers

Using simply the relation that $i^2 = -1$ we can do arithmetic calculations with complex numbers.

- Addition and subtraction are easy: just add or subtract the real and imaginary parts of the complex numbers. For example:

$$\begin{aligned}(3+2i)+(6+7i) &= (3+6)+(2+7)i = 9+13i\\(4-3i)-(\pi-7i) &= (4-\pi)+(-3+7)i = (4-\pi)+4i\end{aligned}$$

  The general rules are these, assuming that a, b, c and d are real numbers (positive, negative or zero):

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i\\(a+bi)-(c+di) &= (a-c)+(b-d)i\end{aligned}$$

- Multiplication is almost like multiplication of binomials in algebra except that additional simplification is possible. For example, if we wish to multiply $3 + 2i$ by $7 + 5i$ we can start by pretending that the i is like a variable x in an algebra expression:

$$(3+2i)\cdot(7+5i) = 21+15i+14i+10i^2 = 21+29i+10i^2.$$

  But now we can simplify this even more, since $i^2 = -1$:

$$21+29i+10i^2 = 21+29i-10 = 11+29i.$$

  When we multiply any two complex numbers, whenever we get an i^2 term, just convert that to -1 and it can be further simplified. In general, again with a, b, c and d being arbitrary real numbers:

$$(a+bi)\cdot(c+di) = (ac-bd)+(ad+bc)i.$$

- Division of complex numbers is possible unless the denominator is zero (meaning that both the real and imaginary parts of the number are zero). If we want to divide $a + bi$ by $c + di$ (where at least one of c or d is non-zero) and put the result in the usual form with a real part and an imaginary part we need to do something akin to the way that we "rationalized" denominators in algebra class.

  If we remember from algebra that $(x + y) \cdot (x - y) = x^2 - y^2$ then we note that if $a + bi$ is a complex number, then so is $a - bi$ and we have:

  $$(a + bi) \cdot (a - bi) = a^2 - (bi)^2 = a^2 - b^2 i^2 = a^2 + b^2,$$

  which is a pure real number (and not equal to zero unless both a and b are zero). Thus here is the general strategy for doing complex division:

  $$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i.$$

  We'll illustrate with one example using actual numbers:

  $$\frac{3 + 4i}{5 - 3i} = \frac{(3 + 4i)(5 + 3i)}{(5 - 3i)(5 + 3i)} = \frac{3 + 29i}{34} = \frac{3}{34} + \frac{29}{34} i.$$

Notice an interesting thing about the powers of i: $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, and so on. As the exponent n in the expression i^n increases, the result simply cycles like this: $1, i, -1, -i, 1, i, -1, -i, \ldots$.

If $z = x + iy$ is a complex number, the complex conjugate of z, written $\overline{z}$ is very useful. The definition is $\overline{z} = x - iy$: it's the same as z except with the opposite sign of the imaginary part. For example: $\overline{3 + 2i} = 3 - 2i$, $\overline{7} = 7$, $\overline{-3i} = 3i$, and $\overline{-6 - 4i} = -6 + 4i$.

If we multiply z by $\overline{z}$ we obtain a (non-negative) real number:

$$z \cdot \overline{z} = (x + iy)(x - iy) = x^2 + y^2.$$

Notice in the calculation of a division above, if we are trying to simplify z/w, where z and w are both complex numbers, we multiply by $\overline{w}/\overline{w}$.

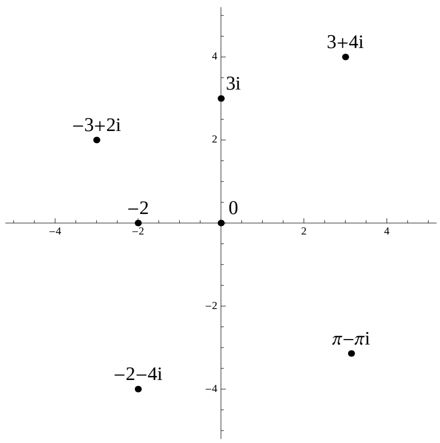
In fact, in complex numbers, we define the absolute value of a number z as follows: $|z| = \sqrt{z\overline{z}}$. If z is real (has no imaginary part) then this definition of absolute value corresponds exactly with the normal one for real numbers: If $z = x + 0i$, then we have $|z| = \sqrt{x^2 + 0^2} = \sqrt{x^2} = |x|$.

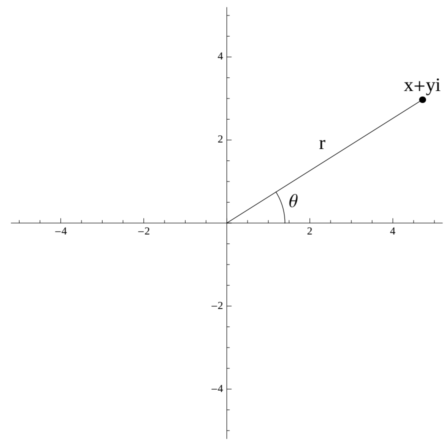
4 Displaying Complex Numbers

The most useful way to visualize complex numbers is as points on a plane. They are plotted in exactly the same way you plot points on a plane with the familiar x- and y-axes, but in this case, what was the x-axis (the horizontal axis) is called the "real axis" and what was the y axis (the vertical axis) is called the "imaginary axis." (In fact, the letter z is often used to represent a complex number, and it is often understood that $z = x + yi$, where x and yi are the real part and the imaginary parts of z. With this assumption the x value goes on what was the x-axis and the y value goes on what was the y-axis.)

The figure on the left below illustrates the positions of a few complex numbers on the complex plane. Their positions are identified in exactly the same way that Cartesian coordinates identify points in the standard x-y plane.

It is also possible to identify points in a plane using so-called "polar coordinates." in the figure below to the right we can see that in addition to the Cartesian version of $x + yi$ we can identify the point by drawing a straight line segment to the point from the origin and listing the length of that segment (r) and the angle that segment makes with the positive horizontal axis (θ). This angle is called the "argument" of a complex number.





It is easy to convert from one set of coordinates to the other:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \mathrm{atan2}(\mathrm{y}, \mathrm{x}) \\ x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

(We use the arctangent function with two variables so that the angle θ will appear in the proper quadrant.)

Note that the Pythagorean theorem tells us that the r in the polar form above is just $|z| = \sqrt{x^2 + y^2} = r$. The following two sections will tell us another way to interpret the θ.

5 A Slight Digression: Taylor Series

Don't worry if you don't know how to derive the following formulas; just assume for now that they're true (which they are):

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \cdots
\end{aligned}$$

In the equations above, $e \approx 2.7182818$ is the base of natural logarithms and the formulas for $\sin$ and $\cos$ assume that x is measured in radians rather than degrees (π radians = $180°$).

These infinite series converge quite rapidly, and we'll demonstrate that by using the one for $\sin$ to approximate $\sin 30°$ which we know to be exactly $1/2$. In radians, $30° = \pi/6$. The table below shows the sums of the series when they are evaluated up to powers of $1, 3, 5, \ldots$, each calculated to 15 places of accuracy. As you can see, the series converges fairly rapidly, and in fact, all three series will eventually converge no matter what value x takes.

n	Sum to n^{th} power
1	0.523598775598299
3	0.499674179394364
5	0.500002132588792
7	0.499999991869023
9	0.500000000020280
11	0.499999999999964

In the same way, we can find an approximate value of e by just substituting $x = 1$ in the series for e^x and obtain: $e \approx 2.71828182845904523536028 74$.

6 Why does $e^{i\pi} = -1$?

You may have seen the equation that's in the title of this section and wondered what in the heck it means. Given the series expansions from the previous section, we can make sense of it. In the expansion of e^x, instead of x, substitute $i\theta$, where θ is a real number and i is the usual (imaginary) square root of -1. We obtain:

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots$$

As we noticed in Section 3, the powers of i cycle and we can write the series above as:

$$e^{i\theta} = 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \frac{(\theta)^4}{4!} + i\frac{(\theta)^5}{5!} - \frac{(\theta)^6}{6!} - i\frac{(\theta)^7}{7!} + \cdots$$

Half the terms above are real and the other half are imaginary, so we can split the expansion above into real and imaginary parts:

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right).$$

But if you compare the series with the real parts and the series with the imaginary parts to the series for $\sin$ and $\cos$ in Section 3, you'll see that they are the same, and we can write:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Since $\cos\pi = -1$ and $\sin\pi = 0$ we obtain $e^{i\pi} = -1$ as a special case, but the general expression for $e^{i\theta}$ is actually far more powerful.

In the figure representing the polar coordinates of a complex number in Section 4 we noted that $x = r\cos\theta$ and $y = r\sin\theta$. In other words:

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

7 Roots of Unity

From elementary algebra, we know that linear equations have a single root and that quadratic equations have two (although they may be the same). It turns out that cubic equations have three roots, quartic equations (equations with 4 as the largest exponent) have four and so on. An n^{th} degree equation has, in general, n roots (some of which may be duplicates). Here we'll consider a more restricted problem. We want to find all the n^{th} roots of 1 (the n^{th} roots of unity). In other words, we want to find all the solutions to the equation:

$$z^n = 1.$$

We already know the answers for $n = 1$ and $n = 2$: If $n = 1$ there is only one root: $z = 1$. If $n = 2$, we have $z = 1$ or $z = -1$. It is also easy to find all the fourth roots of 1: $z = 1, -1, i, -i$. What about when $z = 3$?

If we take the absolute values of all the roots of 1 (unity) we've seen so far, a giant clue for what is going on is the fact that all the absolute values are 1. This means that all the roots have the form $e^{i\theta}$. In fact, here's a list of all the values of θ for roots of 1:

$n = 1$	$\theta = 0$
$n = 2$	$\theta = 0, \pi$
$n = 4$	$\theta = 0, \pi/2, \pi, 3\pi/2$

Notice that in the three cases above, the values of θ that correspond to roots are evenly spaced between 0 and 2π. (Note also that 2π represents a complete loop around the circle and is in fact in terms of direction, the same as an angle of zero.)

If $e^{i\theta}$ is an n^{th} root of 1, then $(e^{i\theta})^n = 1$, so $e^{i(n\theta)} = 1$, so $n\theta$ must effectively be an angle of zero; in other words, an integer multiple of 2π. In the case of the fourth roots above, if we multiply all the values of θ corresponding to the roots by 4 we obtain: $0, 2\pi, 4\pi, 6\pi$.

So to obtain all the cube roots of one, probably all we need to do is to find values of θ which, when multiplied by 3, yield $0, 2\pi$ and 4π. These are obviously: $0, 2\pi/3$ and $4\pi/3$. (In terms of degrees, these are angles of $0°, 120°$ and $240°$.) Using the formula $e^{i\theta} = \cos\theta + i\sin\theta$ we obtain the following three cube roots of 1:

$$z = 1, \qquad z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \qquad z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Just for fun, let's check that $z = -1/2 + \sqrt{3}i/2$ is a cube root of 1. We'll square z and then multiply that square by z again:

$$\begin{aligned} \Big(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\Big)^2 &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ \Big(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\Big)\Big(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\Big) &= 1. \end{aligned}$$

Note two things about this calculation: the square of the first root is the second root. Also, the second root is the complex conjugate of the first. Another way of writing this is as follows. Let ω be the first roots, so $\omega = -1/2 + \sqrt{3}i/2$. Then we have the three cube roots of unity are: $1, \omega$ and $\omega^2 = \overline{\omega}$.

The exact same form works for the other roots of unity we've discovered. In the case of the fourth roots, for example, we have $\omega = i$ and the four fourth roots of 1 are given by: $1, \omega = i, \omega^2 = -1, \omega^3 = -i$.

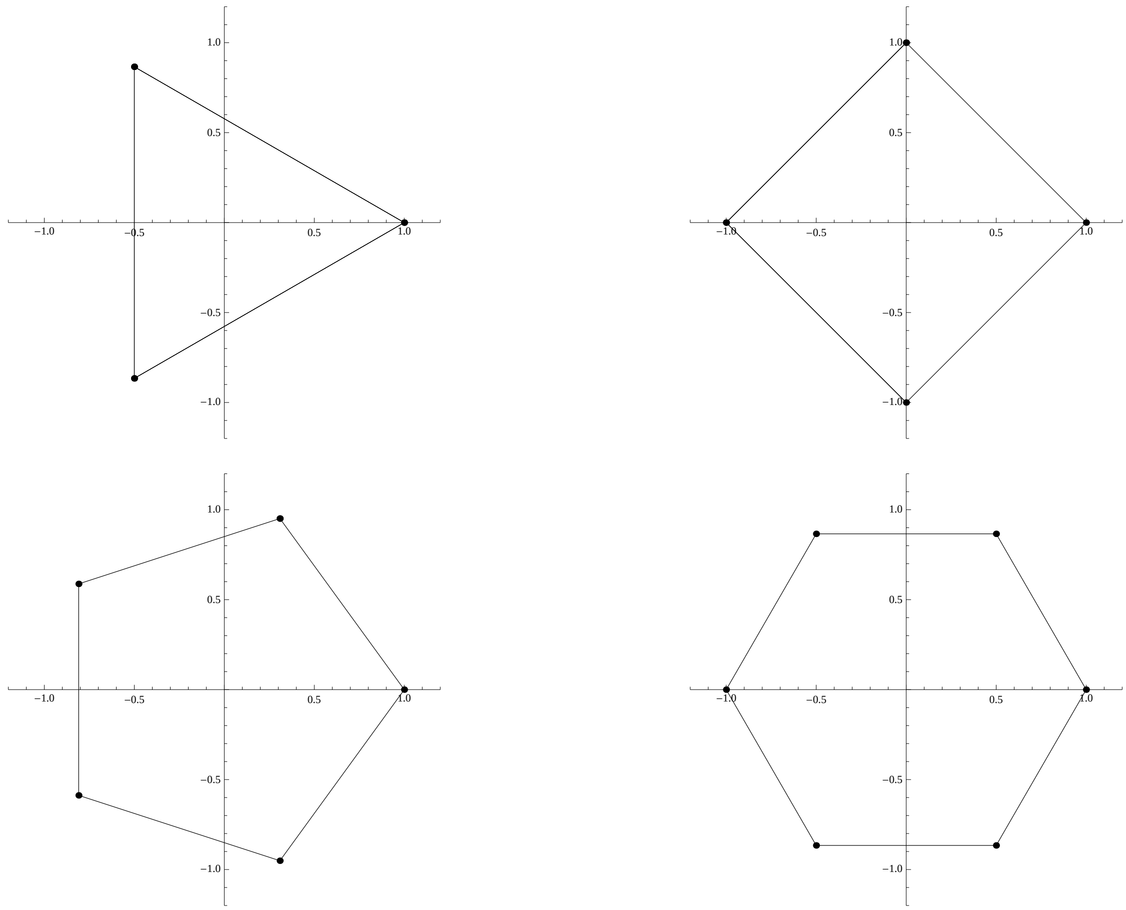
We can do the same thing for any n^{th} roots. Set ω to be $e^{2\pi i/n}$ and take powers of ω for all the roots. If $n = 5$, for example, $\omega = e^{2\pi i/5}$ and the five fifth roots of unity will be: $0, \omega, \omega^2, \omega^3$ and ω^4.

If you're interested in a test of your ability with algebraic manipulations, you can try to convince yourself that the following is the ω corresponding to the fifth root of unity:

$$\omega = \frac{1}{4}\left(-1 + \sqrt{5}\right) + i\sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}}$$

(Hint: You can save one multiplication if you square ω, then square that, and finally multiply the result by ω to obtain ω^5 which should be 1.)

The figures below illustrate exactly what is going on. The points below in the four figures are the cube roots, fourth roots, fifth roots and sixth roots of 1. The lines connecting them are merely to illustrate that they form regular polygons, centered at the origin, and with all vertices a unit distance from that origin:



At the end of Section 2 we asked about the square root of i. The answer is now easy: The number i is the fourth root of 1 so the square root of i will be the eighth root of 1 (or the negative of that). It will therefore be $e^{2\pi i/8} = e^{i\pi/4}$. The angle $\pi/4$ is the same as $45°$, so we have:

$$\sqrt{i} = \frac{1+i}{\sqrt{2}} \quad \text{or} \quad -\frac{1+i}{\sqrt{2}}.$$

8 Roots of Polynomials

In the previous section we found all the solutions to a particular set of polynomial equations:

$$z^n = 1.$$

In this section, we'll see what can be said, usually without proof, about general polynomial equations of the form:

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0 = 0.$$

The standard course of high school algebra completely covers the situation where $n = 1$ or $n = 2$ and where all of the a_i are real numbers. When $n = 2$, quadratic equations have the form:

$$ax^2 + bx + c = 0$$

and the solutions are given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If a, b and c are real, there's no problem if $b^2 - 4ac$ is non-negative: we just obtain the two roots. But if $b^2 - 4ac < 0$ we have to take square roots of negative numbers and consequently find complex roots. Of course if this is the case, it's only the imaginary part that switches sign, so the two complex roots are complex conjugates: if r is one root, the other is $\overline{r}$.

The same sort of thing will happen if all the coefficients of the polynomial are real. If r is a root of the polynomial, then so will be $\overline{r}$. This is easy to prove, once you have convinced yourself of the following facts for any pair $\{z, w\}$ of complex numbers:

$$\overline{z} \cdot \overline{w} = \overline{zw} \quad \text{and} \quad \overline{z} + \overline{w} = \overline{z + w}.$$

The conjugates of all the coefficients are unchanged since they are real, so the conjugate of the polynomial will continue to be zero but with the conjugate of the root in every position.

The general result is this, which is called the "fundamental theorem of algebra":

If a_i are any set of complex numbers such that $a_n \neq 0$, then the polynomial:

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0$$

can be factored as:

$$(z - r_n)(z - r_{n-1})(z - r_{n-2}) \cdots (z - r_2)(z - r_1)$$

so that if it is set to zero, $r_1, r_2, \ldots r_n$ are the n roots of the equation. The r_i are all complex numbers, but there may be any number of duplicates.

This fundamental theorem of algebra is *not* easy to prove.

There exist formulas similar to the quadratic formula that allow us to solve explicitly cubic and quartic polynomial equations, but it is impossible to write an explicit formula for a general polynomial of the fifth or greater degree. This is alson *not* easy to prove.

By the way, the quadratic equation (as well as the formulas to solve cubic and quartic equations) continue to work if the coefficients of the polynomials are complex. As an illustration, let's solve the following quadratic equation:

$$z^2 - (2 + 3i)z - 5 + i = 0.$$

In this case, the a, b and c for the quadratic formula are: $a = 1$, $b = -2 - 3i$ and $c = -5 + i$. Plugging in, we obtain:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 + 3i \pm \sqrt{15 + 8i}}{2}.$$

We can find that $\sqrt{15 + 8i} = 4 + i$, so the roots are $z = 3 + 2i$ and $z = -1 + i$. You can plug these values back into the original equation to check.

How do you find the square root of a complex number? Hint: put the number in the form $z = e^{i\theta}$.

9 Graphing Complex Functions

Life is pretty easy if you want to graph a real function of a single real variable. All the input values can be found along a line or a part of a line, and similarly for all the output values. Thus the entire function can be plotted in two dimensions, where points along the x-axis correspond to inputs and distances in the y-direction correspond to the outputs.

Even real functions of two real variables aren't too difficult to visualize since we have three dimensions available to us: To plot $z = f(x, y)$ (where z is real), choose the input value pairs as points in the x-y plane and the output (z) values are plotted at the appropriate distance above (or below, if z is negative) the x-y plane. The output values will often trace out a surface.

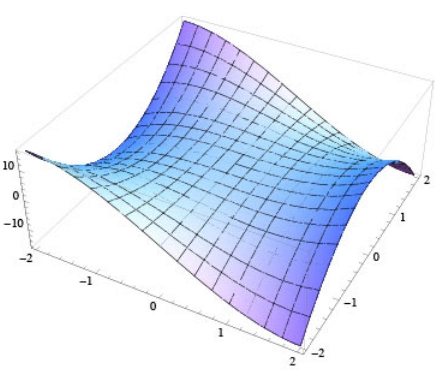
For functions of a complex variable, the situation is more difficult. To draw the function $w = f(z)$ where z and w are complex numbers, both the input and the output are two-dimensional, so we'd need four dimensions to do what we're used to doing with real-valued functions.

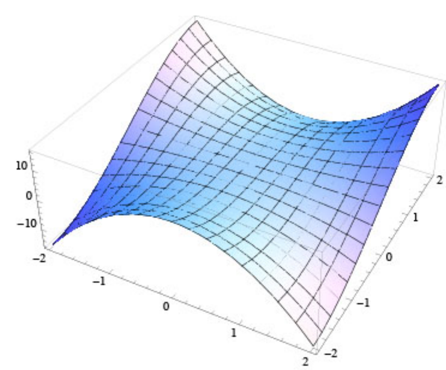
Let's look at a few different ways to graphically illustrate a function. We'll pick a function that is not too hard to understand so it will be easier to see how the visualization corresponds to the function's behavior. Let's use $w = f(z) = z^3$.

If $z = x + iy$ with x and y being real, as usual, then we can write:

$$z^3 = x^3 + 3x^2yi - 3xy^2 - y^3i = x(x^2 - 3y^2) + y(3x^2 - y^2)i.$$

This doesn't seem too useful, but is does indicate a strange symmetry between the real and imaginary parts. We can plot the real part and the imaginary part as a function of z and we have done so below, with $re(z^3)$ on the left and $im(z^3)$ on the right:

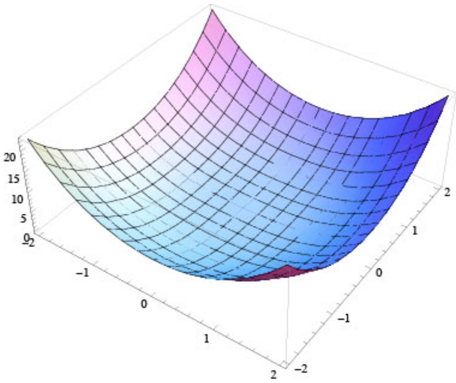


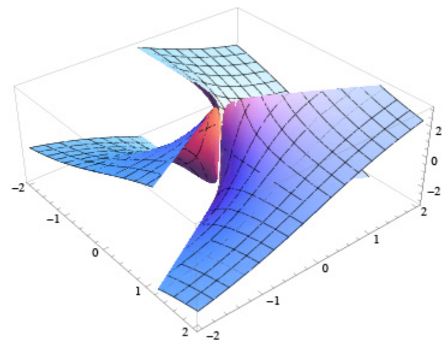


More useful is to write z in polar form: $z = re^{i\theta}$. Then we have:

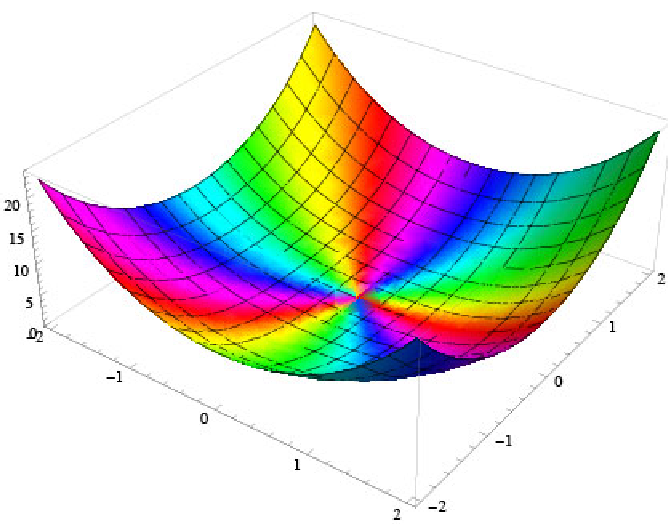
$$z^3 = r^3 e^{i3\theta}.$$

This is nice: the r^3 part is the absolute value of the function and it will depend only on r. The $e^{i3\theta}$ part is the angle at which that modulus will appear. As before, we can plot the r^3 part (the magnitude or absolute value) and the angular part as two graphs below, with the magnitude plot on the left. The plot of the magnitude is not too surprising: it grows like the cube of the radius and is zero at the origin. It is completely symmetric about the origin. The angular part is interesting and appears to be discontinuous. In a sense it is, since when the angle passes π it jumps to $-\pi$ in the same way that if you go past $360°$ you suddenly get to zero, although there's no real discontinuity. The plot just shows that the angle makes three complete loops about the origin.

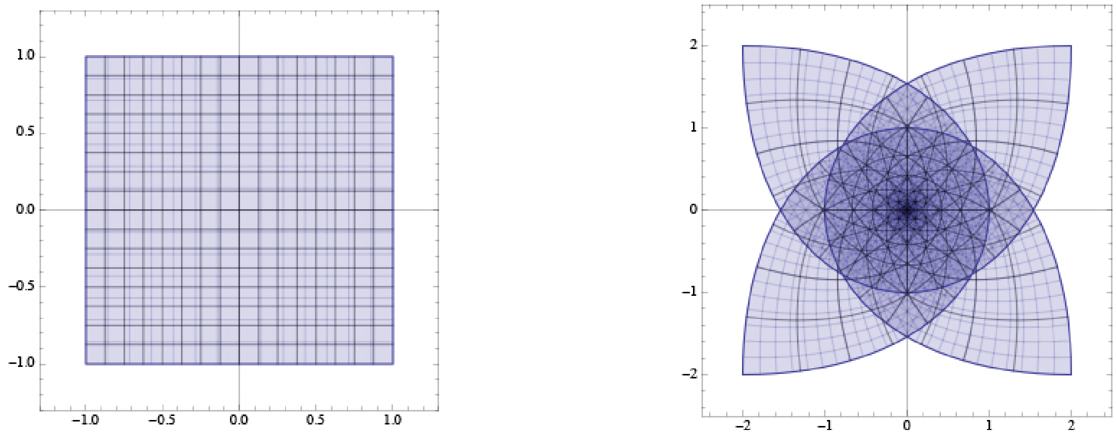




Although we don't have a fourth dimension, we can fake it using colors, especially when the angle is restricted to be between $-\pi$ and π and there's a continuous passage from one to the next. In the image below, we plot the magnitude as before, but we color points on the surface using a loop in the hues of the saturated colors. It's easy to see that the angle loops around three times:



A different way to view it is to label in some way each input point of the function and then attach that same label to where that point is output. One reasonable way to do this is to use a grid to cover the input area and then to see how that grid of lines is mapped by the function. The image on the left below is an input grid covering the area from -1 to 1 in both the real and imaginary directions. The output of that grid is displayed in the image on the right.



The output looks pretty strange since the argument (angle) of the input points is tripled by the function z^3 and since the input image wraps completely around the origin, the output will be wrapped about the origin three times so if we don't do anything special, it will write over itself as it has done here.

For the next example, we'll use a square image (of Professor Paul Zietz) and imagine that it is centered at the origin, and that its coordinates run from -1 to 1 in both the real and imaginary directions. Each pixel in the image corresponds to some complex z value, and wherever $f(z) = z^3$ winds up, paint that z^3 point with the same color as the input point. This will smear Professor Zeitz in some way over the output plane. Here's the input image:

Since we know that the output will wrap itself three times around the origin, what we'll do is to draw three results: the first (on the left) shows only the output from the parts of the input having an argument between $-\pi$ and $-\pi/3$. The next, for inputs between $-\pi/3$ to $\pi/3$ and the final one for inputs between $\pi/3$ and pi.