# Coloring

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## 1 Problems

1. Suppose you have a standard $8 \times 8$ chessboard and a set of dominoes that are exactly the right size to cover two adjacent squares. Two opposite corners of the chessboard are removed. Is it possible to cover the remaining squares using $31$ dominoes?

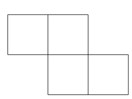
2. (From BAMO 2006) All the chairs in a classroom are arranged in a square $n \times n$ array (in other words, $n$ columns and $n$ rows), and every chair is occupied by a student. The teacher decides to rearrange the students according to the following two rules:

   - Every student must move to a new chair.
   - A student can only move to an adjacent chair in the same row or to an adjacent chair in the same column. In other words, each student can move only one chair horizontally or vertically. (Note that the rules above allow two students in adjacent chairs to exchange places.)

   For which values of $n$ is this possible?

3. Suppose, instead of dominoes, you have "trominoes": sets of three squares attached together in a straight line. If you try to cover a full $8 \times 8$ chessboard it obviously cannot be done, since $3$ does not divide evenly into $64$. The best result you can hope for is to use $21$ trominoes and cover $63$ of the $64$ possible squares, leaving one square uncovered. Can such a covering be achieved for an arbitrary square chosen as the one to be left uncovered? If not, for which omitted squares is it possible to achieve a covering of all the others using trominoes?

4. (From BAMO 2008) This time, consider a $9 \times 9$ chessboard, and you wish to cover as much of it as possible using figures shaped like the one below (which we will call a "tetromino"), where each of the four squares is the same size as the squares on a chessboard. The pieces can be rotated or flipped over. What is the maximum number of non-overlapping pieces that you can fit?



5. The Cheese Cube. Suppose that a cube of cheese is sliced into $27$ sub-cubes (so that it looks something like Rubik's cube). A mouse starts eating in one corner and eats each sub-cube completely before eating an adjacent sub-cube. (An adjacent sub-cube is one that shares a face with the previous sub-cube.) Can the mouse find a path so that the last cube he eats is the one in the center?

6. The four-color theorem states that any planar map of countries or states can be colored with four different colors in such a way that no two countries that share a boundary are colored with the same color. Of course the countries have to be connected, and countries that share just a single point are not assumed to share a boundary: the boundary has to have non-zero length. This theorem was not proved for many years, but was finally proved by Appel and Haken in 1989 with the help of computers.

Our problem is easier and does not require hundreds of hours of computer time. Suppose that the "world" is a plane, and that a number of circles are drawn on the plane. These circles may be of different sizes, and they may overlap, but the net result is that they finally divide the plane into a number of regions. Suppose that each region is a country, and find the minimum number of colors required to color them satisfying the same conditions as in the original four-color theorem.

7. In a group of six people, every pair of people have either shaken hands or they have not. Show that there is a set of three people such that either:

   - All three have shaken hands with the other two.
   - Or, none of the three have ever shaken hands.

   Is this true for every group of five people?

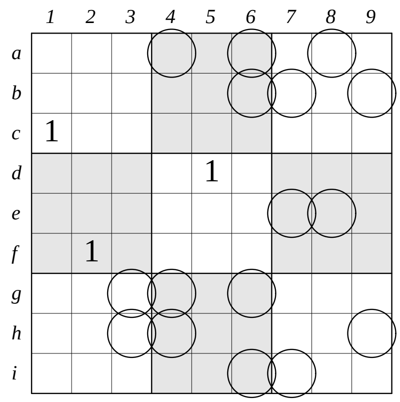
8. Coloring in Sudoku. A Sudoku puzzle consists of a $9 \times 9$ grid divided into $3 \times 3$ sub-blocks with some of the grid squares containing numbers between $1$ and $9$. A completed puzzle contains the numbers $1$ through $9$ arranged in some order in the grid so that each row, column and sub-block contains each of the numbers exactly once. To solve a puzzle, you must begin with the initial partially-filled grid and complete it as described above. The figure below shows a puzzle on the left and the solution on the right.

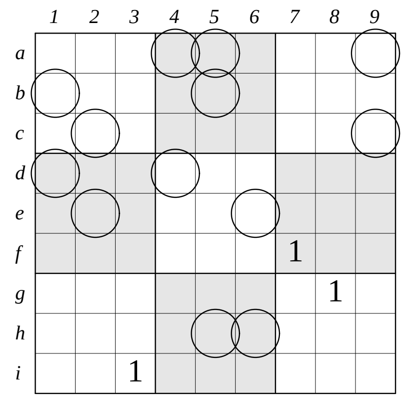
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| a | | | 4 | 8 | | | | | |
| b | | 9 | | 4 | 6 | | | 7 | |
| c | | 5 | | | | | 6 | 1 | 4 |
| d | 2 | 1 | | 6 | | | 5 | | |
| e | 5 | 8 | | 7 | | 9 | | 4 | 1 |
| f | | | 7 | | | 8 | | 6 | 9 |
| g | 3 | 4 | 5 | | | | | 9 | |
| h | | 6 | | | 3 | 7 | | 2 | |
| i | | | | | | 4 | 1 | | |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| a | 6 | 2 | 4 | 8 | 7 | 1 | 9 | 5 | 3 |
| b | 1 | 9 | 3 | 4 | 6 | 5 | 8 | 7 | 2 |
| c | 7 | 5 | 8 | 3 | 9 | 2 | 6 | 1 | 4 |
| d | 2 | 1 | 9 | 6 | 4 | 3 | 5 | 8 | 7 |
| e | 5 | 8 | 6 | 7 | 2 | 9 | 3 | 4 | 1 |
| f | 4 | 3 | 7 | 1 | 5 | 8 | 2 | 6 | 9 |
| g | 3 | 4 | 5 | 2 | 1 | 6 | 7 | 9 | 8 |
| h | 8 | 6 | 1 | 9 | 3 | 7 | 4 | 2 | 5 |
| i | 9 | 7 | 2 | 5 | 8 | 4 | 1 | 3 | 6 |

   A sudoku puzzle can be viewed as a coloring problem. Imagine that all $81$ squares in the grid are vertices of a (mathematical) graph. The vertices are connected with edges if they lie in the same row, column, or sub-block. The original numbers in the puzzle are nine different "colors" the goal is to color the graph in such a way that no two vertices connected by an edge share the same color.

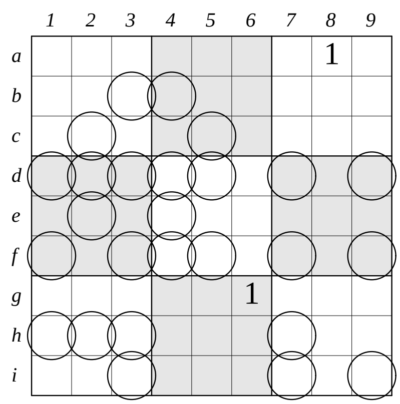
   In the following two grids, suppose you are concentrating on which squares contain the number $1$. In both cases, the squares containing the $1$ are shown, and the squares that are circled indicate the only other possible squares where a $1$ could be placed. Every unfilled square that is not circled must contain a number other than $1$. Can you make any additional conclusions in the two puzzles about which squares either must or must not contain a $1$?

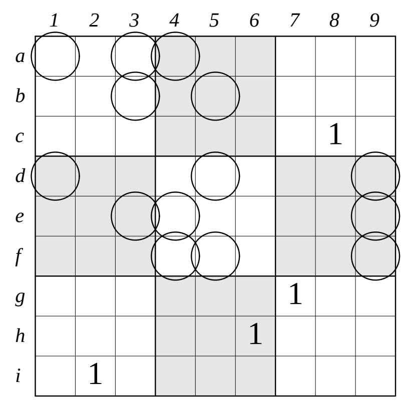




9. Multi-Coloring in Sudoku

   This is a similar idea that of simple coloring used in the previous example, but involves two (or more) chains of colors. Here are two examples like those above, where the number under consideration is 1 and the circled squares indicate possible positions for the remaining 1 entries. What conclusions can you draw?





# 2 Solutions

1. Suppose that the chessboard is colored normally with alternating white and black squares. Each domino, no matter where it is placed will cover one black and one white square. Originally, the chessboard will contain $32$ white and $32$ black squares. If the two corners are removed, then you have removed two squares of the same color so the resulting chessboard will have $30$ squares of one color (say white) and $32$ of the other (black). At most $30$ dominos can be placed since each covers one white square, so it is impossible to cover the entire chessboard which would require $31$ dominoes.

2. This problem is very similar to the previous one. Color the chairs alternately, as you would a chessboard, with white and black. As described in the problem, for a switch of the students to occur, each student in a black chair has to move to a white chair and vice-versa. Thus there must be the same number of white and black squares for a seat-switch to be possible. If $n$ is odd, then there are $n^2$ seats, which is also odd, so there cannot be the same number of white and black squares. Thus it is impossible in this case.

   It is easy to construct a swapping pattern if $n$ is even: starting from one end, swap students in adjacent columns of seats with the student in the other column but in the same row.

3. To solve this problem we need to color the board with three colors, which we'll call $1$, $2$ and $3$. Consider the two colorings shown below:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 |
| 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 |
| 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 |
| 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 |
| 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 |
| 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |

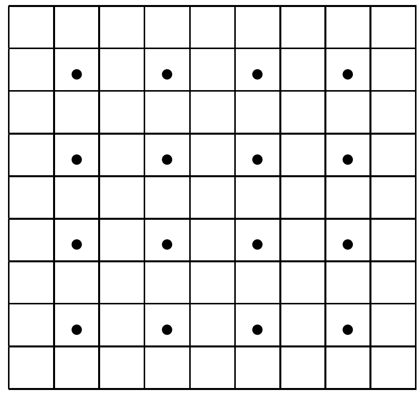
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 |
| 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 |
| 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 |
| 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 |
| 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 |
| 3 | 1 | 2 | 3 | 1 | 2 | 3 | 1 |

   Clearly each tromino covers a $1$, $2$ and $3$ on both boards. On the board on the left, there are $21$ $1$'s, $22$ $2$'s and $21$ $3$'s. This means a square marked $2$ must be left uncovered. In the board on the right, there are $22$ of the $1$'s (and $21$ of the $2$'s and $3$'s), so a $1$ must be left uncovered on that board. There are only four positions that are covered by a $2$ on the first board and a $1$ on the second. The possible empty squares are illustrated in the following figure:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| | | | | | | | |
| | | • | | | • | | |
| | | | | | | | |
| | | | | | | | |
| | | • | | | • | | |
| | | | | | | | |
| | | | | | | | |

   This is not a complete proof: we need to demonstrate with at least one tiling that the board can be covered with one of the squares marked with a dot above left uncovered. If we can do one, the four rotations of that board will provide the other three examples. This is easy to do, however.

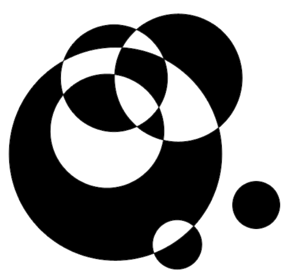
4. This time, consider the following “coloring” where the dotted squares have one color and the non-dotted squares, another:



It is easy to see that no matter how the tetromino is placed on the board, it will cover exactly one of the black dots. Since there are exactly $16$ black dots, the largest number of tetrominoes possible is $16$. It is easy to place $16$ non-overlapping tetrominoes on the board to show that $16$ is possible as well as maximal.

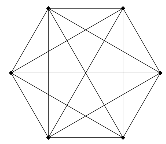
5. This is impossible. Color the sub-cubes alternately as you would a three-dimensional chessboard. If a corner is colored black, then there will be $14$ black sub-cubes and $13$ white ones. Any mouse path must alternate black and white cubes. If the mouse eats all $27$ sub-cubes, the final one will have to be black, but the sub-cube in the center is white.

6. This is fairly easy to prove. Imagine that the plane is initially colored entirely one color, say white. Now, every time you draw a circle, reverse all the colors of the regions inside the new circle. This shows that any such map can be colored with exactly two colors. The following figure illustrates the idea.

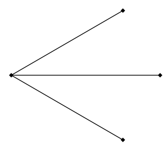


7. We will show that this is true by considering a “handshake graph”. Each of the vertices of the following graph represents one of the six people, and the edge between any pair of vertices will be represented by a line. In this paper, all the edge lines are black, but what we would like to do is to color the edge red if the two particular people at the ends of the edge have shaken hands, and we will color it blue if that particular pair of people have not shaken hands. Thus, no matter what the situation, all of the edges in the graph will be colored, with some of them red and the rest of them blue.

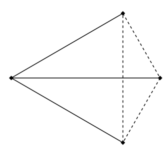
   The situation that we would like to prove exists is that no matter how this coloring is done, there is either a triangle somewhere in the figure with all three of its edges red, or a triangle somewhere in the figure with all of its edges blue. (Of course there could be multiple triangles like this, or both red and blue triangles. All that matters is that there be at least one triangle of a solid color.)



Suppose, on the contrary, that there is no solid red triangle and no solid blue triangle. Since every vertex on the graph has $5$ edges going out of it, choose a particular vertex and at least three of the edges coming out have the same color. Suppose that color is red, and the reverse argument will work if it is blue. In the figure below, the three solid edges coming from the vertex at the left represent the three red lines.



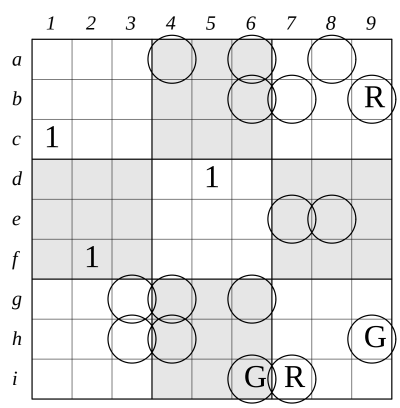
We know that there are no red triangles, so the three vertices on the right cannot be connected with red edges, but they must be connected, so all the connecting lines must be blue, yielding the following figure, where the blue lines are dashed:



But this yields a solid blue triangle connecting all the vertices on the right. Thus it is impossible not to have some triangle of a solid color.

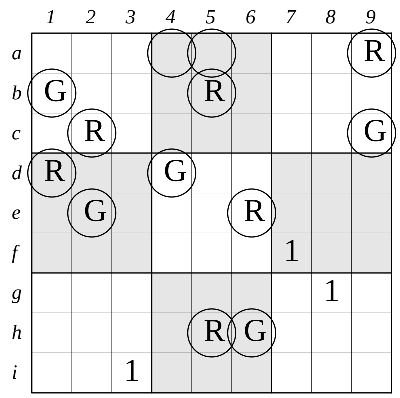
8. Notice that in Sudoku it is impossible to have the same number appearing twice in the same row, column or block. This means that if there are exactly two squares in a row, column or block that could possibly contain that number, exactly one of them will and the other will not. Thus we could color one red and one green, and if the red contains the number, the green will not, and vice-versa. Suppose you have colored two such squares, but the red square appears in a different row, column or block with one other possible square. Then that square can be colored green, et cetera, forming red-green "chains". As you walk along a chain, alternate colored squares appear, and in the final solution, either all the red squares or all the green squares will contain that candidate number and all the squares of the opposite color will not.

   If we color the first example as illustrated below, with "R" for "red" and "G" for "green", we obtain:



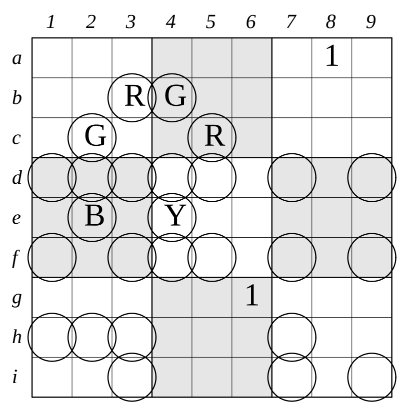

   Now consider the square at $b6$. It is in the same row as a red square and in the same column as a green square. Either the red square or the green square must contain a $1$, so the square at $b6$ can *not* contain a $1$. If the circle around $b6$ is erased, there are suddenly only two possibilities in the $b$ row, so the square at $b7$ can be colored green, and using the same argument, the square at $e7$ cannot contain a $1$. That, of course, forces $e8$ to contain a $1$, and at least one more elimination can be made.

Much more can be done with the second example. Here it is after some of the squares are colored red and green as before, where we begin by coloring $a9$ as red:



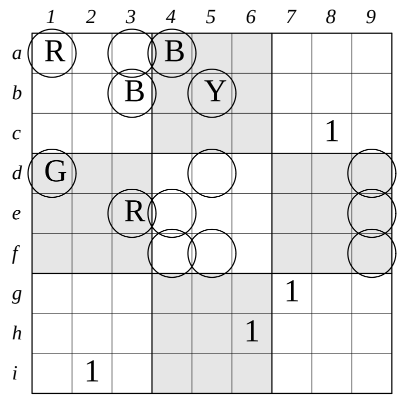

Almost all the squares can be colored, but there is a surprising result: two spots in the fifth column are red and since they cannot both contain a $1$, neither must, so all the green squares must contain a $1$.

9. Multicoloring is just an extension of simple coloring, but where multiple chains of colors are possible. In the first example, we have colored two chains: a red-green (R-G) chain and a blue-yellow (B-Y) chain.




Notice that if green holds the $1$, then neither blue nor yellow can hold a $1$. But one of blue or yellow *must* hold a $1$. Therefore the red squares contain the $1$.

Now let'slook at the second example, again colored with a red-green and a blue-yellow chain:




Since the red and blue appear in the same line (and in the same block), it cannot be true that both the red and blue cells contain a $1$. That means that either the green or the yellow cells contain a $1$, so there is a $1$ in either $d1$ or $b5$. Thus there cannot be a $1$ in $d5$.

We can build a formal mathematical system to deal with the interactions of colored chains in a multicolored Sudoku puzzle. Rather than use color names, let's suppose that each chain consists of colors labeled with pairs of uppercase and lowercase letters. Thus we will have chains of "colors" $a$ and $A$, of "colors" $b$ and $B$, of $c$ and $C$, and so on. All the chains, of course, deal with the same candidate number. In all of the examples that follow, let's assume that the candidate in question is $1$.

Think of it this way: "$a$" stands for the logical proposition: "If all the squares marked with $a$ hold a $1$, then none of the squares marked with $A$ contain a $1$, and vice-versa: If all the squares marked with $A$ contain a $1$ then none of the squares marked with $a$ contain a $1$. If "$\neg$" stands for the logical operator "not", then we can express the relationships above as "$a \equiv \neg A$" or equivalently, "$A \equiv \neg a$". The "$\equiv$" stands for "is logically equivalent to".

Now suppose we've marked as many squares that have $1$ as a possible candidate with chains of $a - A$, of $b - B$, of $c - C$, and so on, where each is as long as possible. (By the way, these are not strictly "chains", since there may be branching.)

What happens if squares marked $a$ and $b$ appear in the same row, column or block? The expression we will use here when that occurs is "$a$ excludes $b$". In other words, if $a$ is true, then $b$ must be false, and also, if $b$ is true, then $a$ must be false. We will write $a!b$ to mean "$a$ excludes $b$".

The "!" or "excludes" logical operator is symmetric; in other words, $a!b \equiv b!a$.

The expression $a!b$ can also be stated, "at most one of $a$ or $b$ is true". It could be that neither is true, but if one of them is true then the other must be false. If you are familiar with logic gates in electrical engineering, this is the "nand" logical operator which stands for "not and": "$a$ nand $b$" means "it is *not* the case that both $a$ and $b$ are true, and both may be false".

After you have marked the complete color chains in any Sudoku puzzle, you can then find pairs that exclude each other because they collide on a line or in a block. You might have, for example, a set like this of exclusions:

$$\{a!b, c!D, d!A, C!e\}. \qquad (1)$$

The nice thing about this notation is that sometimes pairs of exclusions can imply other pairs. The general rule is this: If $a!b$ and $B!c$ then $a!c$. In English, "If $a$ excludes a color, and the opposite of that excluded color excludes a third color $c$, then $a$ excludes $c$." See if you can prove this before continuing.

It is not too hard to prove. The expression $a!b$ means that if $a$ is true then $b$ cannot be true, so if $a$ is true, then the opposite of $b$, which is $B$ must be true. But since $B!c$ this means that $c$ cannot be true. Put these together and conclude that if $a$ is true, $c$ cannot be true. Reverse the logic staring with $c$: if $c$ is true, $B$ cannot be true. Thus $b$ is true, and hence $a$ cannot be true, so the truth of $c$ implies the falsity of $a$. This is equivalent to $a!c$. Notice that we used the symmetry of the "!" operator: that $a!b \equiv b!a$ and $B!c \equiv c!B$.

If the sample set in Equation 1 applied to a particular puzzle, from $a!b$ and $d!A$ we can conclude that $a!d$, and from $c!D$ and $C!e$ we can conclude that $D!e$. Make sure you see why. By using this exclusionary version of transitivity, we can discover exclusionary features of chains that do not interact directly with each other, but only indirectly, via other chains.

So fine, we can list exclusionary color chain relations and possibly find more such exclusionary relations, but how can we use these to either eliminate the candidate $1$ from some Sudoku squares or better yet, determine that some squares must contain the candidate $1$?

Here is how to use them. If $a!b$, then at least one of $A$ or $B$ must be true (since if neither $A$ nor $B$ is true then both $a$ and $b$ are true which is impossible). Thus if $a!b$ and we find a square that shares a row, column or block with both an $A$ and a $B$, that square cannot contain the candidate $1$.

Let's try applying what we have to the first multicoloring example presented here. We used $G - R$ and $B - Y$ as our chains, so to use the same notation, let $G - R$ be replaced by $a - A$ (in other words, $G$ is $a$ and $R$ is $A$. Similarly, let $B - Y$ be replaced by $b - B$. In this example, we have: $a!b$ and $a!B$. We combine these two to conclude that $a!a$. This means "either $a$ or $a$ is false", or equivalently, "$a$ is false". So $A$ is true.

Using the same mappings as above to $a - A$ and $b - B$ in the second multicoloring example, we have: $A!b$. This means that any candidate square that shares a row, column or block with both $a$ and $B$ cannot contain a $1$. The square $d5$ is such a square and thus can be eliminated as a candidate.

The following example will allow you to experiment with multicoloring. The candidate in this case is, as always, $1$. Assume that due to various considerations, the only squares that can contain candidate $1$ are marked with a small $1$ in the upper left corner. Multicolor the puzzle below and see what conclusions can be drawn.

| | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | *9* |
|---|---|---|---|---|---|---|---|---|---|
| *a* | 4 | 8 | $^{1}$ | $^{1}$ | 6 | $^{1}$ | 2 | 7 | 5 |
| *b* | 2 | 5 | | | 7 | | 1 | | 6 |
| *c* | $^{1}$ | 7 | 6 | | $^{1}$ | $^{1}$ | 4 | 3 | |
| *d* | 5 | 2 | $^{1}$ | 8 | 4 | $^{1}$ | 6 | $^{1}$ | 3 |
| *e* | $^{1}$ | 6 | 8 | 3 | 5 | 2 | 7 | 4 | $^{1}$ |
| *f* | 3 | $^{1}$ | $^{1}$ | $^{1}$ | 9 | 6 | 8 | 5 | 2 |
| *g* | 8 | 3 | 5 | | $^{1}$ | $^{1}$ | 9 | 6 | 7 |
| *h* | 6 | 9 | $^{1}$ | | 8 | | 3 | $^{1}$ | 4 |
| *i* | 7 | $^{1}$ | $^{1}$ | 6 | 3 | 9 | 5 | $^{1}$ | $^{1}$ |

# 3 Sudoku Coloring/Multicoloring Examples

Below are six Sudoku examples where coloring or multicoloring can be applied. In the examples below, try to apply coloring (or multicoloring) using candidate 6 on the left and 2 on the right in the first row. In the second row, consider squares where 4 is a candidate in both, and in the final row, use 7 on the left and 4 on the right.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| a | 3 | 9 | (2 6) | 8 | 7 | 5 | 4 | (2 6) | 1 |
| b | 8 | 5 | 4 | (3 9) | (2 6) | 1 | (2 3 9) | 7 | (2 3 6 9) |
| c | 7 | (2 6) | 1 | (3 9) | (2 6) | 4 | 8 | 5 | (3 9) |
| d | 6 | 4 | 8 | 2 | 5 | 3 | 1 | 9 | 7 |
| e | 2 | 3 | 7 | 4 | 1 | 9 | 6 | 8 | 5 |
| f | 9 | 1 | 5 | 7 | 8 | 6 | (2 3) | 4 | (2 3) |
| g | 5 | (2 6 8) | 9 | 1 | 3 | (7 8) | (2 7) | (2 6) | 4 |
| h | 4 | 7 | 3 | 6 | 9 | 2 | 5 | 1 | 8 |
| i | 1 | (2 6 8) | (2 6) | 5 | 4 | (7 8) | (2 7 9) | 3 | (2 6 9) |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| a | 7 | 9 | (5 8) | 6 | (3 5) | (4 8) | (3 4) | 2 | 1 |
| b | 1 | 6 | (2 5) | 7 | (3 5) | (2 4) | (3 4 9) | (3 8) | (4 8 9) |
| c | (2 4) | 3 | (2 4 8) | (1 9) | (2 8) | (1 9) | 7 | 6 | 5 |
| d | (4 6) | 7 | (1 3 9) | 2 | (1 4 9) | 5 | (1 3 6 9) | (3 8) | (8 9) |
| e | (2 6) | 5 | (1 3 9) | 8 | (1 9) | 7 | (1 2 3 6 9) | 4 | (2 9) |
| f | 8 | (2 4) | (1 9) | 3 | (1 4 9) | 6 | (1 2 9) | 5 | 7 |
| g | 3 | 8 | 7 | (4 5 9) | 6 | (2 9) | (2 4 5) | 1 | (2 4) |
| h | 5 | (2 4) | 6 | (1 4) | 7 | (1 2 8) | (2 4 8) | 9 | 3 |
| i | 9 | 1 | (2 4) | (4 5) | (2 8) | 3 | (2 4 5 8) | 7 | 6 |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| a | 3 | (2 9) | 7 | 8 | 6 | 5 | 4 | (1 2) | (1 9) |
| b | (8 9) | 6 | (4 8) | 3 | (2 4) | 1 | (7 9) | (2 7) | 5 |
| c | 5 | (2 4) | 1 | (2 9) | 7 | (4 9) | 8 | 6 | 3 |
| d | 4 | 8 | 9 | (2 5) | (2 5) | 6 | 1 | 3 | 7 |
| e | 7 | 5 | 3 | 4 | 1 | 8 | 6 | 9 | 2 |
| f | (2 6) | 1 | (2 6) | 7 | 9 | 3 | 5 | 4 | 8 |
| g | (2 6 8 9) | 7 | (2 4 6 8) | (1 6 9) | (4 8) | (4 9) | 3 | 5 | (1 6 9) |
| h | 1 | (3 4 9) | (4 6) | (5 6 9) | (3 4 5) | 7 | 2 | 8 | (6 9) |
| i | (6 8 9) | (3 9) | 5 | (1 6 9) | (3 8) | 2 | (7 9) | (1 7) | 4 |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| a | 2 | 9 | (3 7) | 8 | (6 7) | (5 6) | (4 5) | (3 4 5) | 1 |
| b | (5 8) | 6 | 4 | 3 | (1 2) | (1 5) | (5 8 9) | 7 | (2 5 9) |
| c | (5 8) | (1 3 7) | (1 3 7) | 9 | (2 7) | 4 | (5 8) | 6 | (2 3) |
| d | (1 4 7) | (1 2 4 7) | 9 | 6 | 5 | 3 | (1 4 7) | (1 2 4) | 8 |
| e | 3 | (2 7) | 8 | 1 | 4 | 9 | (5 7) | (2 5) | 6 |
| f | 6 | (1 4) | 5 | 2 | 8 | 7 | 3 | (1 4 9) | (4 9) |
| g | (1 4) | 8 | (1 3 6) | 7 | (1 3 6 9) | 2 | (1 4 5 6 9) | (1 3 4 5 9) | (3 4 5 9) |
| h | (1 7) | 5 | (1 3 6 7) | 4 | (1 3 6 9) | (1 6) | 2 | 8 | (3 9) |
| i | 9 | (1 3 4) | 2 | 5 | (1 3 6) | 8 | (1 4 6) | (1 3 4) | 7 |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| a | (2 8) | 9 | 6 | (7 8) | 5 | 4 | 3 | (2 7 8) | 1 |
| b | 7 | 5 | (4 8) | 3 | 2 | 1 | 9 | (4 8) | 6 |
| c | (1 2 3 4 8) | (2 3 4 8) | (1 2 4 8) | 9 | (7 8) | 6 | (4 7 8) | 5 | (2 4 7 8) |
| d | 6 | (2 7 8) | 9 | 4 | 1 | (7 8) | 5 | 3 | (2 7 8) |
| e | 5 | (2 4 7 8) | 3 | (2 7 8) | 6 | 9 | 1 | (2 4 7 8) | (2 4 7 8) |
| f | (2 4 8) | 1 | (2 4 8) | (2 7 8) | 3 | 5 | (4 6 7 8) | (2 4 6 7 8) | 9 |
| g | (1 3 4 8) | 6 | (1 4 8) | 5 | (7 8) | 2 | (4 7 8) | 9 | (3 4 7 8) |
| h | (2 3 4 8) | (2 3 4 8) | 7 | 1 | 9 | (3 8) | (4 6 8) | (4 6 8) | 5 |
| i | 9 | (3 8) | 5 | 6 | 4 | (3 7 8) | 2 | 1 | (3 7 8) |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| a | 8 | 9 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| b | 6 | 5 | 4 | 3 | 2 | 1 | 9 | 8 | 7 |
| c | 2 | 3 | 1 | 9 | 8 | 7 | 6 | (4 5) | (4 5) |
| d | 1 | 8 | 9 | 4 | 7 | 3 | 5 | 6 | 2 |
| e | 3 | (4 7) | 5 | 2 | 9 | 6 | 1 | (4 7) | 8 |
| f | (4 7) | 6 | 2 | 5 | 1 | 8 | (4 7) | 3 | 9 |
| g | (5 9) | (2 4) | 8 | 7 | 3 | (5 9) | (2 4) | 1 | 6 |
| h | (5 7) | (2 7) | 6 | 1 | 4 | (2 5) | 8 | 9 | 3 |
| i | (4 9) | 1 | 3 | 8 | 6 | (2 9) | (2 4 7) | (4 5 7) | (4 5) |