Mathematical Card Tricks

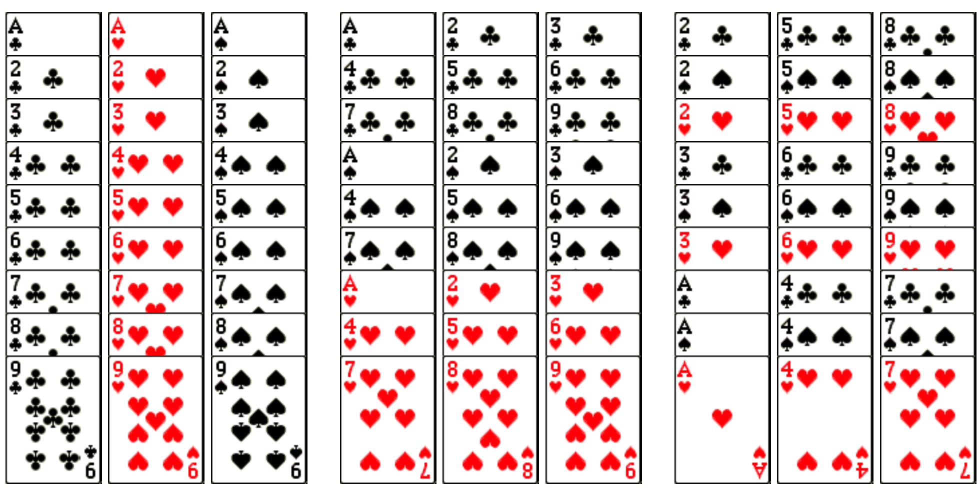
Tom Davis
tomrdavis@earthlink.net
http://www.geometer.org/mathcircles
May 26, 2008

At the April 10, 2008 meeting of the Teacher's Circle at AIM (the American Institute of Mathematics) I presented a few card tricks that are based almost entirely on mathematics. A couple of other, similar tricks were presented, and I will include those as well.

1 Base 3 Arithmetic

Trick: Deal 27 cards, face up onto a table in three columns with 9 cards in each. Ask your victim to choose a card but not to tell you what it is. Three times, ask in which column the card lies, and each time, collect the cards, putting the selected column between the cards in the other two columns, leaving the columns in order, and then deal the cards again by rows to make the next set of columns. After the third choice has been made and the cards re-dealt for the fourth time, the selected card will be number 14 – exactly half-way through the deck.

Discussion: Each choice by the victim reduces the number of possibilities by 1/3. Since there were 27 choices originally, there will be 9 after the first selection, 3 after the second, and 1 after the third. You can show what is happening by starting with each column consisting of a different suit, and it's even better if the cards in the columns are in order, say A, 2, 3, ..., 9. After the first selection, the selected suit must be in the middle three rows. The next selection will make the selected card be one of those in the middle row, and the final selection will force it to be in the exact middle: card number 14.



The three deals above show what happens if you begin with the cards in the arrangement on the left, and the victim's card happens to be a spade. The club row is put on top of the spade row and that on top of the heart row, so the second arrangement shows the cards after the second deal. All spades are in the middle three rows, in order.

Next, assume that the victim's card is in the third column (and so it must be a 3, 6 or 9, and we also know it was a spade). That column is put between the other two and dealt out, and as you can see, the 3, 6 and 9 of spades are in the center positions of each column. When the victim chooses his final column, and it is put between the other two, it will be the fifth card in the middle column, and will thus have $9 + 4 = 13$ cards on top of it, making it the fourteenth card.

Note that after the third column selection is made, there is no need to re-deal the cards; you know the selected card is in the middle of the selected column, and once you know the card, you can do anything else in the world with the deck: shuffle it, add more cards from the rest of the deck, et cetera. Note also that if you make different, but consistent, choices of where to insert the selected column, you can make the chosen card appear in any position. For example, if you place the selected column on top each time, after the third choice, the card will be on the very top of the pile.

This can be thought of as numbering the cards in base 3, starting with zero:

000, 001, 002, 010, 011, 012, 020, 021, 022, 100, 101, 102, ..., 221, 222

If you think of placing the selected column on top, you add a zero; the middle is a 1, and the bottom is a 2, so if you place the selected column first on the bottom, then in the middle, then on top, the card will appear in position 210 = the twenty-first position (base 10).

One easy way to add some excitement to the presentation is that after you have collected the cards for the third time (and you know that the selected card is number 14), as you deal out the cards, deal 15 of them face-up, so you have actually gone one past the victim's chosen card. At this point, bet that the next card you turn over will be his. He, of course, sees that you passed his card already, and will be certain you are wrong. After he makes a bet, turn the fourteenth card over, so it is face-down.

2 Magic

Trick: Deal three piles of three cards face-down. Have the victim choose a pile and turn it over to view the bottom card on the pile. Place the selected pile on top of the other two and spell out the name of the card as follows.

Suppose the card is the 4 of spades. Spell out the words: "FOUR" "OF" "SPADES". With each letter, deal a card from the top of your original pile face down onto a new pile. When the letters run our, drop the rest of the remaining cards, in order, on top of the cards you spelled out. Do this three times, for each of the three words in the card's name. Finally, spell out the word "MAGIC", where the first four cards, corresponding to the letters "MAGI" are face down, but deal the final one, corresponding to the "C" IN "MAGIC", face up. That will be the selected card.

Discussion: This works since the names of the cards have between 3 and 5 letters, no matter what the rank of the card: "ACE", "TWO", ..., "JACK", "QUEEN", "KING". That means that since the selected card was the third one down, no matter what its name, it will be third from the bottom after the spelling of the rank. Spelling out "OF" always moves it two up in the deck, so it will be the middle card (fifth from the bottom). The suit names all have between 5 and 8 letters: "CLUBS", "DIAMONDS", "HEARTS" or "SPADES", thus on the final spelling, the the fifth card (which is the selected card) will again be in position 5, no matter what. And the fifth card is the one selected by the spelling of "MAGIC".

A good way to demonstrate this is to make the selected card a different color so the students can see how it moves through the deals.

3 Parity

I learned this trick from Art Benjamin at the eighth Gathering for Gardner.

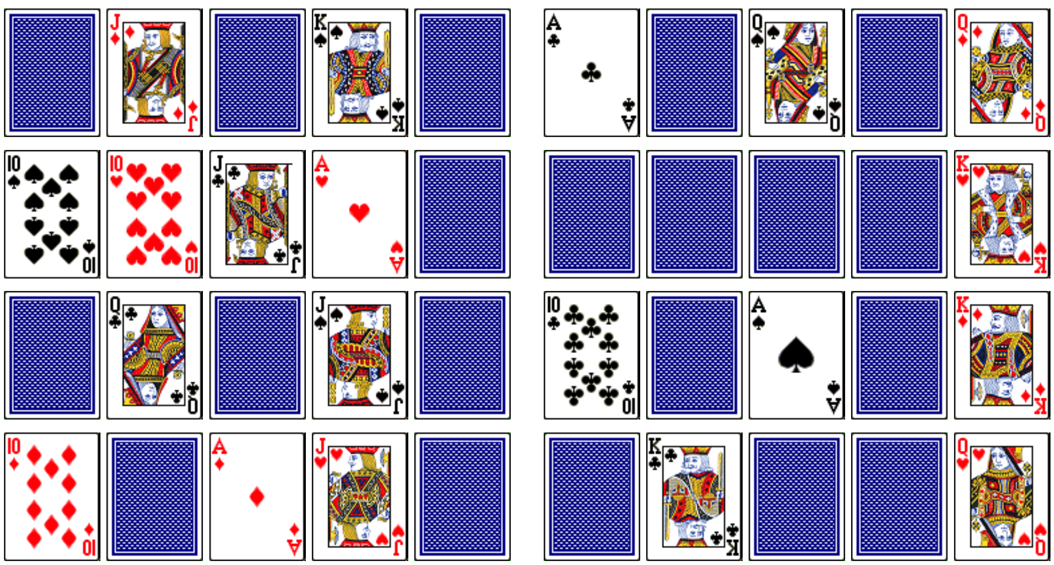
Trick: Select just the cards from 10 to ace in each suit for a total of 25 cards. Ask the victim for his favorite suit and shuffle the cards as much as desired. Next, go through the deck picking two cards at a time. After each pair is chosen and arranged, ask the victim whether to place it on a new pile as-is, or if he wants you to turn it over. Do what he says in every case.

If neither of the two cards is the favorite suit, place them face-together in a new pile. If both are in the favorite suit, place them facing away from each other in the pile. If there is one from the favorite suit and one that's not, there are two cases. If the favorite suit is the top card, leave them as-is, and ask about flipping the pair or not before placing it on the pile. If the favorite suit is on the bottom, here's where a tiny amount of slight-of-hand is required. The instant you see that the cards are in the "wrong" order separate them to show them to the victim. When you put them back together, he probably won't notice that you reversed the order so that the card of the favorite suit is on top. At this point, ask whether to flip the cards or not.

After all ten pairs, or twenty cards, have been arranged in a pile as in the previous paragraph, take them four at a time and place those in a new pile, asking whether to flip them all or leave them as-is. Next you are going to deal rows of five cards to make a rectangular arrangement of the 20 cards, but before you deal each row of five, ask the victim whether to begin dealing from the left or the right. Do what he says.

Now you have a rectangular array of $5 \times 4 = 20$ cards and tell the victim that you want to fold them together, but for each fold, it's his choice whether to fold from the top, bottom, left or right. A "fold" consists of taking each pile in the top, right, left or bottom row, depending on what the victim chose, and flipping it over onto the pile below, to the left, to the right, or above, respectively. The "piles" initially consist of a single card, but after each fold, some of the piles will have more than one.

Finally, the entire set of 20 will be folded into a single pile. You can see, assuming that the pile is a bit irregular, whether you can see a card of the favorite suit facing up or whether you see one of a non-favorite suit facing up. If you see a non-favorite suit, pick up the pile and turn it over; otherwise, just pick it up. Now spread out the cards, and you'll see that all and only those cards from the favorite suit are face-up.



Discussion: This is a parity-based trick. After the first arrangement in sets of two, notice that if we count the cards starting from one at the bottom of the arranged set of pairs that if a card of the favorite suit is in an odd position, it is facing down, and if it is in an even position, it is facing up.

When you then take sets of four cards, whether you flip them or not, the same situation continues to hold.

Next, imagine that you are dealing the cards into four rows of five on top of a checkerboard of alternating colors. The odd cards will wind up in positions 1, 3 and 5 in the first and third rows, and in positions 2 and 4 in the second and fourth. In other words, if the checkerboard had black squares in the top right and left, then every card of the favorite suit will be face-down on the black squares and face-up on the red ones, and the opposite holds for all the cards in the non-favorite suits. During any fold, cards from red squares are turned over on top of black ones and vice-versa, so at the end of the folding, all the cards in the favorite suit will be facing the same direction and all non-favorites face in the opposite direction. We can't control whether the favorites wind up facing up or down, since that depends on the order the victim chose for the folds.

The illustration above is an example where the victim's favorite suit is hearts. The set of 20 cards on the left is the original deal, and the set on the right is the same cards, but every one is turned over so you can see the other side. Notice that the face-up hearts are only in "black" checkerboard squares, and are face-down (in other words, face-up on the right) in the "white" checkerboard squares. Similarly, all the non-hearts are face-up on the "white" squares on the right and face-down on the "black" checkerboard squares.

Note that after you've got the final folded pile, you can shuffle it, if you like, since that won't turn cards upside-down. Similarly, when you've got the grid of 20 cards, you can offer the victim the choice of exchanging any two cards on "red" checkerboard positions or any two in "black" positions. Or any other parity-preserving operation you can think of.

To show the kids how it's working, exchange all the cards in the favorite suit for similar cards with a different-colored back and at each stage you can show how the parity conditions are preserved.

4 Predicting a Card

Note: this trick does not always work, but it works about 85% of the time. If you can do it with two decks, it will work about 98% of the time. Also, this takes a slightly more sophisticated victim than the previous ones. The victim needs to be able to count without errors. **As presented in class, this trick only works about** $2/3$ **of the time.**

Trick: The victim chooses a number (say between 1 and 10), and then you start at the beginning of a shuffled deck (or better, a shuffled pair of decks) and you deal the cards face up, slowly, one at a time. As you deal, the victim silently counts the cards until his number of cards is dealt. In other words, if he chose 5, he will silently count the cards you deal as $1, 2, 3, 4, 5$. When his card (the fifth in this example) is reached, he looks at the rank of that card which will be from ace $(= 1)$ to king. If the card is a face card (jack, queen or king), count its rank as 5 (**This is not what we did in class**). As soon as he sees that card, the rank is his new number, and he begins counting from one again beginning with the next card you turn up.

Repeat this process, and each time he reaches his count, that card's rank is his new number. Finally, as he is counting up the deck will run out, and he will not have gotten to his number. Whatever card he was using as his latest number is his final card. At this point, you tell him what that card is.

As a concrete example, suppose the beginning of the deck looks like this, where we use A, J, Q and K to stand for "ace", "jack", "queen" and "king", respectively. Since the suits don't matter for the counting, they are not listed below:

$$3, A, 4, 2, Q, Q, K, 3, 8, 3, 5, 6, 6, 10, J, A, A, 5, 8, 4, 9, 5.$$

If the victim's initial secret number is 3, he will count, "1, 2, 3", as the 3, ace, and 4 are dealt. His card, the third, is the 4, so at that point his new number is 4, and as he counts to 4, the cards 2, queen, queen and king are dealt. The fourth card is the king, so his new number is 5. He counts from one to five, which will be a 6, then he will get a 5 and since there are only 4 remaining cards in the deck, his final card will be 5.

Using the same example above, figure out what your final number would be (assuming the list above represented the whole deck) if your initial number were 2. Do this now.

OK, your answer should be 5 – in fact the same 5 as you got starting with an initial number of 4. Now try an initial number of 10. You should find that your final card is 8.

In fact, it's probably best to go through the deck once starting with some particular number and counting aloud to show the victim exactly how the counting is done.

When you do the trick for real, though, you can almost always figure out the victim's card! The way to do this is just to pick your own starting number and do the exact thing that the victim is doing. About 85% of the time, you'll wind up at the same final card.

Discussion: The reason this works is that as you're both going through the deck, if you ever happen to hit the same card as the victim, from then on you will both be locked into the same "path" through the rest of the deck and will certainly arrive at the same final card. With a double deck, you just have about twice as many chances of hitting the same card the victim does during the counting.

If you do the trick as presented in class, where jack, queen and king represented the numbers 11, 12, and 13, then you'll only win about $2/3$ of the time.

You can make a mathematical estimate as follows. The "average" advance will be by:

$$(1 + 2 + 3 + \cdots + 9 + 10 + 5 + 5 + 5)/13 = 5.38$$

cards, so you will take, on average, about $52/5.38$, or about 8 steps. The density of the victim's cards is about $1/5.38 = .186$, so there's a $1 - .186 = .814$ chance of a miss. But you have to miss all 8 times, and this will occur about $(.814)^8 = .174$ so you'll win about $.836$ of the time.

I wrote a computer program to simulate this, and after a million trials, the winning percentage was $.84169$. If jack is 11, queen 12 and king 13, the winning percentage goes down to $.68284$ (again with a million trials). Finally, if we just spell out the names of the cards, so "ace" advances us by three cards, et cetera, I simulate the probability to be $.95347$. With two decks, the percentages are, respectively, $.97708$, $.90219$ and $.99823$.

To make it much more likely to win, have the victim spell out the name of the card, as "ACE", for three steps, et cetera. The final section contains the simulator code that I used to obtain these numbers.

This trick is apparently due to the physicist Martin Kruskal. Look up "Kruskal count" or "Kruskal card trick" for more information.

5 Psychic Card Guessing

This one is due to Josh Zucker.

Trick: You place a red and a black ace face-up, and tell the victim that he's psychic, and can tell the color of the cards as you deal them, even though you don't expose any of them. You also have the other two aces face-up to the side.

As you pick each card off the top of the deck, but while it is still face-down, ask the victim to tell you the color. If he says "red", put it on the pile under the red ace, and if black, on the pile with the black ace on top. After a while, you stop dealing, and place the remaining red ace face-up on the black pile and the black ace face-up on the pile that had the red ace. Ask the victim to continue until you get to the end of the cards. At this point, ask the victim to see how he did, and you flip over one of the piles and by golly, under the red ace are all red cards and under the black one are all black cards. After the victim has marveled at this for a while, show him that the same thing happened with the other pile.

Discussion: You need a stacked deck for this one. In fact, it's easier not to use the entire deck, but just, say, 20 cards: 10 black and 10 red. To stack it, just put all the red cards first, then all the black ones. As you deal the cards, placing them wherever you are told, count up to 10, and when the tenth card is dealt, place the other aces on the piles, so that you now know that the pile with the red ace will now have a black ace on top, and since the last ten cards are black, anything put on this pile will work.

Of course the other pile is messed up and while the victim is amazed and distracted by the fact that he got every card right in the first pile, you need to pick up the other deck and secretly swap the aces. Then it will be perfectly right, too. Alternatively, you can just show it as-is, and be amazed that although the victim got them 100% right in the first pile, for some reason he was 100% wrong on the second.

6 Matching Cards

This is due to Brian Conrey.

Trick: This isn't really a trick, but is a good bar-bet, or sucker-bet, where the result is not intuitive. Take two full decks of cards, shuffle them, and ask your victim to deal cards face-up slowly, one at a time, at the same time you do the same with the other shuffled deck. You bet your victim that there will be at least one exact match in the dealt cards. "Exact" means both suit and rank. If you are betting money and can get an even bet, you will win about $2/3$ of the time, and will make money in the long run. What's surprising is that the length of the deck doesn't make much difference after it is five or six cards long or longer. The odds are still close to $2/3$ that there will be a match.

Discussion: This is a bit beyond the scope of this paper, but it is based on the mathematical topic of "derangements" that you can investigate on the internet. In fact, Imagine that Yankee Stadium sells all its seats, but the $50,000$ customers walk in and select seats at random, without looking at their tickets. The odds are still about $2/3$ that at least one customer will wind up in his correct seat, even with a "deck" of $50,000$.

7 Guessing the Fifth Card

This is known as the "Fitch Cheney" trick, and is fairly sophisticated. It requires an accomplice who is also good. The victim, however, need not be sophisticated at all.

Trick: The victim selects any five cards from a deck of cards while the performer is not present and gives them to the accomplice. The accomplice picks a single card from the set, places it face-down, and hands the four remaining cards to the performer when she returns. The performer looks at the four cards and names the face-down card. In fact, the accomplice can give the pack of four cards to the victim to hand to the performer and leave the room to assure that he is not giving the performer any information by body-language, et cetera. All the performer has to work with is the deck of four cards.

Discussion: On the face of it, this is impossible. The performer sees only four cards which could be arranged in $4! = 24$ different orders, but there are $52 - 4 = 48$ possibilities for the remaining face-down card. But there is more: the accomplice can not only arrange the deck of four cards, but can choose the face-down card, so he actually has $5 \times 4! = 120$ different courses of action. There are many ways to do it, but what follows is a reasonable approach that does not require too much mental effort (although it does require some, and definitely requires practice, especially to do it quickly).

First, the accomplice and the performer need to agree on an ranking for a deck of cards. Any ranking will do, and if they are bridge players, for example, they might agree that any 2 is less than any 3, and so on, up to jack, queen, king, and ace is the "largest". If there are two equal cards, the "bridge ordering" of the suits determines the larger card:

$$\text{clubs} < \text{diamonds} < \text{hearts} < \text{spades}.$$

Since there are five cards, at least two of them must be in the same suit, and if you agree to count in a circle, as: $2, 3, 4, \ldots, 9, 10$, jack, queen, king, ace, $2, 3, \ldots$, then starting from one of those two cards you can get to the other in six or fewer steps. For example, if the two cards are the 4 and king, then the king is the starting card, and counting up as: "ace, $2, 3, 4$" makes the 4 exactly four steps up from the king.

The accomplice places this starting card on top of the deck of four cards to be handed to the performer, so the performer immediately knows the suit and a starting place. To determine the card, she only needs to know how many steps up to take to get there, and this number will be between 1 and 6. Luckily, there are six permutations available for the remaining three cards, and the performer and accomplice need only agree on an assignment of orderings to numbers to pass this information using the remaining three cards. The three cards are ranked as discussed above, so assume that A, B and C represent the smallest, middle, and largest card, respectively, using that ranking. Here's a possible assignment:

$$1 = ABC \quad 2 = ACB \quad 3 = BAC \quad 4 = BCA \quad 5 = CAB \quad 6 = CBA.$$

This is relatively easy to remember since the rank orders are in alphabetical order.

As an example, suppose the five cards chosen by the victim are $2S$, $2H$, QH, $5D$ and $2C$, where "QH" means the "queen of hearts", et cetera.

The only pair of cards that are in the same suit are the $2H$ and the QH. It's too many steps from the two to the queen, so the queen of hearts must be the starting card, and there are three steps necessary to get to the two from the queen: "king, ace, 2". Thus the accomplice places the queen of hearts on the top of the deck of four cards and places the two of hearts face-down on the table. The remaining cards are ordered as follows: $2C < 2S < 5D$ and we need to order them to encode "three steps". Three is encoded by "middle, low, high", or $2S, 2C, 5D$, so the deck handed to the performer looks like: $QH, 2S, 2C, 5D$.

8 Monte Carlo Simulator Code

Here's the code used to obtain the probability estimates in Section 4.

```
#include <stdlib.h>
#include <stdio.h>

#define DECKS 2

int CARDS = 52*DECKS;
int deck[52*DECKS];


int DEBUG = 0;

// card i has rank 1+(i/4) and suit i%4.

void testprint()
{
    int i;
    for (i = 0; i < CARDS; i++)
        printf("%2d ", rank(deck[i]));
    printf("\n");
}

void init()
{
    int i, j;
    for (j = 0; j < DECKS; j++)
        for (i = 0; i < 52; i++) deck[i + 52*j] = i;
}

int spellrank[13] = {3, 3, 5, 4, 4, 3, 5, 5, 4, 3, 4, 5, 4};

int rank(int card)
{
    return spellrank[card/4]; // "ace" = 3, "two" = 3, et cetera
    int i = card/4+1;
    if (i > 10) i = 5;
    return i;
}

void shuffle()
{
    int i, j, tmp;
    for (i = CARDS-1; i > 0; i--) {
        j = rand() % i;
        tmp = deck[i];
```

```
        deck[i] = deck[j];
        deck[j] = tmp;
    }
}

main(int argc, char **argv)
{
    int count = atoi(argv[1]);
    int match = 0, nomatch = 0;
    int i, card, pos, mycard;
    init();
    for (i = 0; i < count; i++) {
        shuffle();
        if (DEBUG) testprint();
        pos = rand() % 10; // victim's choice
        if (DEBUG) printf("%d ", pos);
        while (pos < CARDS) {
            card = deck[pos];
            pos += (rank(card));
            if (DEBUG) printf("%d ", pos);
        }
        if (DEBUG) printf("\n");
        pos = rand()%10; // my choice
        if (DEBUG) printf("%d ", pos);
        while (pos < CARDS) {
            mycard = deck[pos];
            pos += (rank(mycard));
            if (DEBUG) printf("%d ", pos);
        }
        if (DEBUG) printf("\n");
        if (card == mycard) match++; else nomatch++;
    }
    printf("match: %d, nomatch: %d, percent: %g\n",
                        match, nomatch, 1.0*match/count);
}
```