Geometry (Mostly Area) Problems
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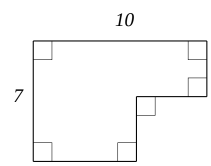
Instructions: Solve the problems below. In some cases, there is not enough information to solve the problem, and if that is the case, indicate why not. The problems are not necessarily arranged in order of increasing difficulty. Many of the problems below are modified versions of problems from various AJHSME (American Junior High School Mathematics Examination) contests collected by Tatiana Shubin. If there is a (*) or (**) in front of a question, that problem (or parts of it) are more difficult. Two stars is harder than one star, et cetera.

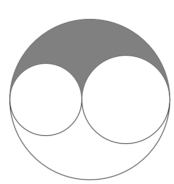
1 Easier Problems

1. **(AJHSME, 1986)** Find the perimeter of the polygon below.

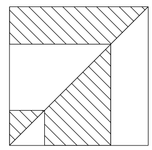


2. **(AJHSME, 1986)** (*) In the figure to the left below, the two small circles both have diameter 1 and they fit exactly inside the larger circle which has diameter 2. What is the area of the shaded region? In the figure on the right, again assume that the larger circle has radius 2 and the three smaller circles are equal. What is the area of the shaded region? What if there were n small circles (arranged in the same way) and the large circle still has radius 2? What is the perimeter of each of the regions? What happens to the area and perimeter of an n-circle drawing as n gets large? (As a mathematician would say, "…as n approaches infinity?")

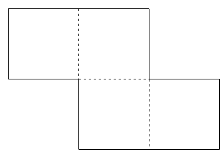
3. (*) What is the area and perimeter of the shaded region in the figure below? This time assume that the two smaller circles have different diameters, and that the larger circle has diameter equal to 2.



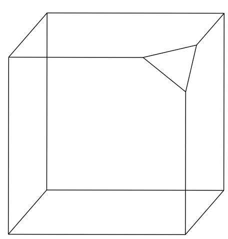
4. **(AJHSME, 1990)** The square below has sides of length 1. Other than the diagonal line, all the other lines are parallel to the sides of the square. What is the area of the shaded region?



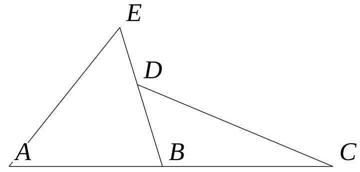
5. **(AJHSME, 1990)** The figure below is composed of four equal squares. If the total area of the four is 100, what is the perimeter? If the perimeter is 100, what is the area?



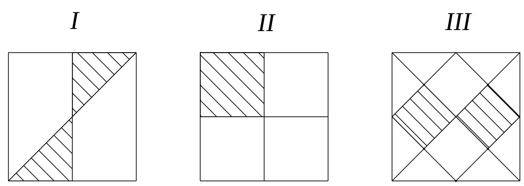
6. **(AJHSME, 1990)** Suppose that all of the corners of a cube are cut off (the figure below illustrates the operation when a single corner is cut off). How many edges will the resulting figure have?



7. **(AJHSME, 1994)** In the figure below (which is *not* drawn to scale), we have $\angle A = 60°$, $\angle E = 40°$, $\angle C = 30°$. Find the measure of $\angle BDC$.

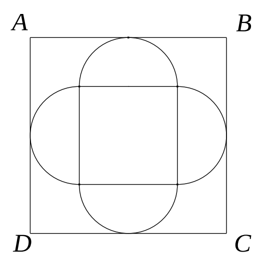


8. **(AJHSME, 1994)** All three squares below are the same size. Compare the sizes of the shaded regions in each of the three squares.

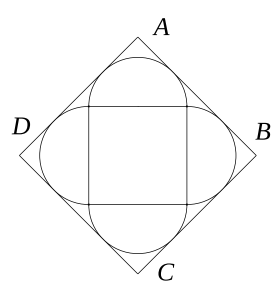


9. **(AJHSME, 1994)** (*) The perimeter of a square is 3 times the perimeter of another square. What is the ratio of the areas of the squares? If the surface area of a cube is 3 times the surface area of another cube, what is the ratio of their volumes?

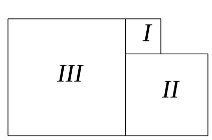
10. **(AJHSME, 1994)** The inner square in the figure below has area 3. Four semicircles are constructed on its sides as shown below. A new square, $ABCD$, is constructed tangent to the semicircles and parallel to the sides of the original square. What is the area of square $ABCD$?



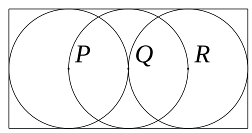
11. (**) The inner square in the figure below has area 3. Four semicircles are constructed on its sides as shown below. A new square, $ABCD$, is constructed tangent to the semicircles and perpendicular to the diagonals of the original square. What is the area of square $ABCD$?



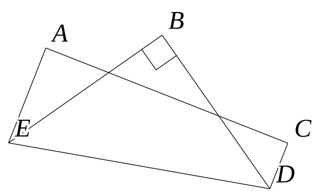
12. **(AJHSME, 1995)** In the following figure (not drawn to scale) the perimeter of square I is 10 and the perimeter of square II is 20. What is the perimeter of square III? What is the combined perimeter of the figure formed from all three squares?



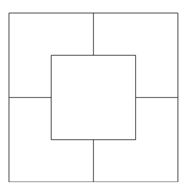
13. **(AJHSME, 1995)** Three congruent circles with centers P, Q and R are tangent to the sides of the rectangle in the figure below. The circle centered at Q has diameter 4 and passes through points P and R. What is the area of the rectangle?



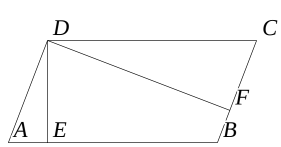
14. **(AJHSME, 1995)** In the figure below, $\angle A$, $\angle B$ and $\angle C$ are right angles. If $\angle AEB = 40^\circ$ and $\angle BED = \angle BDE$ then what is the measure of $\angle CDE$?



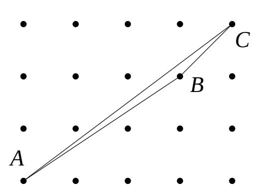
15. **(AJHSME, 1995)** In the figure below, the large square is 100 inches by 100 inches. Each of the congruent L-shaped regions account for $3/16$ of the total area of the square. What are the dimensions of the inner square?



16. **(AJHSME, 1995)** In parallelogram $ABCD$ below DE and DF are the altitudes from D to sides AB and BC, respectively. If $DC = 12$, $EB = 4$ and $DE = 6$ then $DF =$?

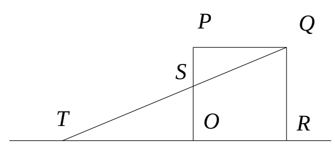


17. **(AJHSME, 1996)** In the rectangular grid below, the distances between adjacent vertical and horizontal points equal 1 unit. What is the area of the triangle $\triangle ABC$?

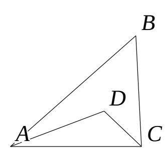


18. **(AJHSME, 1996)** Points A and B are 10 units apart. Points B and C are 4 units apart. Points C and D are 3 units apart. If A and D are as close as possible, then what is the number of units between them?

19. **(AJHSME, 1996)** In the figure below, O is the origin, P lies on the y-axis, R lies on the x-axis and the point Q has coordinates $(2, 2)$. What are the coordinates of T such that the area of triangle $\triangle QRT$ is equal to the area of the square $OPQR$? If S is the intersection of TQ and PO, what are the coordinates of T such that the area of triangle $\triangle TSO$ is equal to the area of $OPQR$?

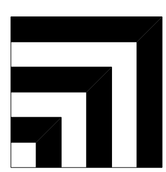


20. **(AJHSME, 1996)** In the figure below, $\angle B = 50°$. The line AD bisects $\angle CAB$ and CD bisects $\angle BCA$. Find the measure of $\angle ADC$.

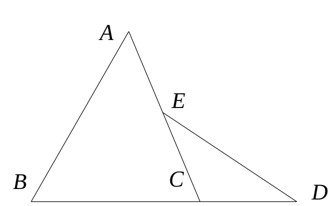


21. **(AJHSME, 1997)** What is the area of the smallest square that will contain a circle of radius 4? What is the area of the largest square that will be inside a circle of radius 4?

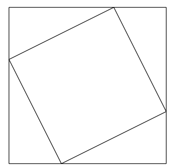
22. **(AJHSME, 1997)** In the figure below, what fraction of the square region is shaded? Assume that all the stripes are of equal width.



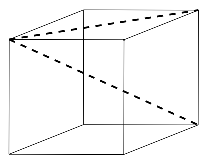
23. **(AJHSME, 1997)** In the figure below, $\angle ABC = 70°$, $\angle BAC = 40°$ and $\angle CDE = \angle CED$. Find the measure of $\angle CED$.



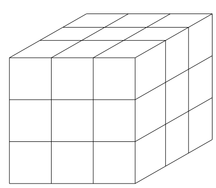
24. **(AJHSME, 1997)** In the figure below, the sides of the larger square are trisected and a smaller square is formed by connecting the points of trisection as shown. What is the ratio of the area of the inner square to the area of the outer square?



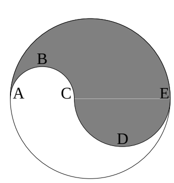
25. **(AJHSME, 1997)** The cube in the figure below has eight vertices and twelve edges. The segments shown with dashed lines that connect two of the cube vertices but are not edges are called diagonals. How many diagonals does the cube have?



26. The cube below is 3 cm $\times$ 3 cm $\times$ 3 cm and is formed of 27 identical smaller cubes. Suppose that a corner cube is removed. What is the surface area of the remaining figure? What if all eight corner cubes are removed? What if all the cubes in the centers of the faces are removed (and the corner pieces remain)?

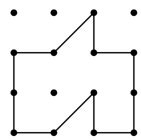


27. **(AJHSME, 1997)** The diameter ACE of the circle below is divided at C in a ratio of $2:3$. The two semicircles ABC and CDE divide the circular region into an upper (shaded) region and a lower region. Find the ratio of the area of the upper region to that of the lower region.

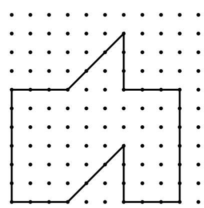


28. **(AJHSME, 1998)** In the figure on the left below, how many triangles are there? (Some triangles may overlap other triangles.) How about in the figure on the right?

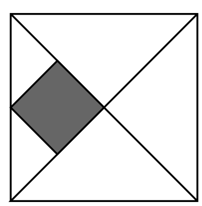
29. **(AJHSME, 1998)** In the figure below dots are spaced one unit apart, vertically and horizontally. What is the number of square units enclosed by the polygon?



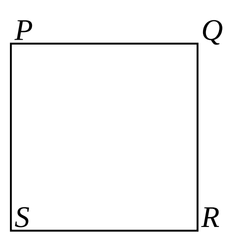
30. In the figure below dots are spaced one unit apart, vertically and horizontally. What is the number of square units enclosed by the polygon?



31. **(AJHSME, 1998)** In the figure below find the ratio of the area of the shaded square to that of the large square.



32. **(AJHSME, 1998)** In the figure below let $PQRS$ be a square sheet of paper. P is folded onto R and then Q is folded onto S. The area of the resulting figure is 9 square inches. Find the perimeter of the original square $PQRS$.

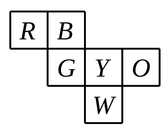


33. **(AJHSME, 1998)** A $4 \times 4 \times 4$ cubical box contains 64 identical small cubes that exactly fill the box. How many of these small cubes touch a side or the bottom of the box?

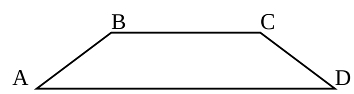
34. **(AJHSME, 1999)** What is the degree measure of the smaller angle formed by the hour and minute hands of a clock at 10 o'clock?

35. **(AJHSME, 1999)** A rectangular garden 50 feet long and 10 feet wide is enclosed by a fence. To make the garden larger, while using the same fence, its shape is changed to a square. By how many square feet does this enlarge the garden?

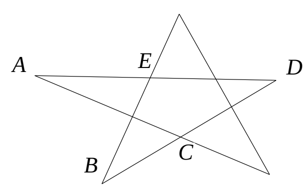
36. **(AJHSME, 1999)** In the figure below six squares are colored, front and back, (R = red, B = blue, O = orange, Y = yellow, G = green, and W = white). They are hinged together as shown, then folded to form a cube. What color is the face opposite the white face?



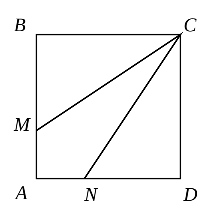
37. **(AJHSME, 1999)** In the figure below $ABCD$ is a trapezoid with $AD = 16$, $BC = 8$, the altitude is 3, and $AB = CD$. Find the perimeter of $ABCD$.



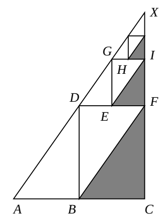
38. **(AJHSME, 1999)** In the figure on the below $\angle B = 40°$, $\angle BED = 100°$ and $\angle ACD = 110°$. Find the measure of angle A in degrees.



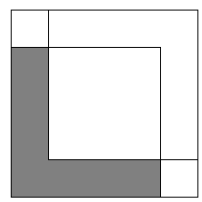
39. **(AJHSME, 1999)** In the figure below $ABCD$ is a square with sides of length 3. The segments CM and CN divide the square into three regions that have equal areas. How long is the segment CN?



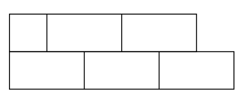
40. **(AJHSME, 1999)** In the figure below $\triangle ACX$ is a right triangle with $\angle ACX = 90°$. Points B, D and F are the midpoints of the sides of $\triangle ACX$. Similarly, points E, G and I are the midpoints of the sides of $\triangle DFX$. Continue dividing and shading in the same way 100 times. Assume $AC = 6$ and $CX = 8$. To three decimal places, what is the sum of the areas of all 100 shaded triangles.



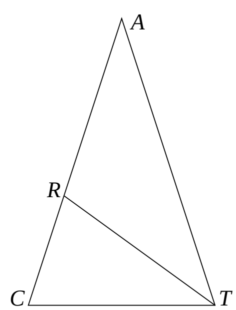
41. **(AJHSME, 2000)** In the figure below a 3×3 square is centered in a 5×5 square with the sides parallel as shown. Find the area of the shaded region.



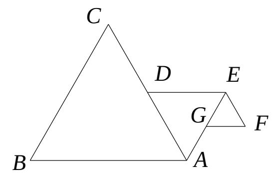
42. **(AJHSME, 2000)** A block wall 100 feet long and 7 feet high will be constructed using blocks that are 1 foot hand and either 2 feet or 1 foot long (no blocks may be cut). The vertical joins in the blocks must be staggered as in the figure below and the wall must be even on the ends. What is the smallest number of bricks required to build this wall? What is the largest number?



43. **(AJHSME, 2000)** In triangle $\triangle CAT$ below we have $\angle ACT = \angle ATC$ and $\angle CAT = 36°$. If TR bisects $\angle ATC$ then what is the measure of $\angle CRT$?



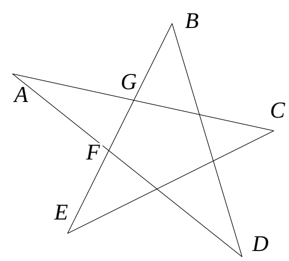
44. **(AJHSME, 2000)** In the figure on the below triangles $\triangle ABC$, $\triangle ADE$ and $\triangle EFG$ are all equilateral. Points D and G are the midpoints of AC and AE, respectively. If $AB = 4$, what is the perimeter of the figure $ABCDEFG$?



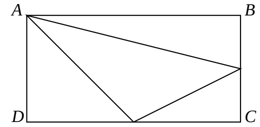
45. **(AJHSME, 2000)** In order for Martin to walk a kilometer (1000 meters) in his rectangular backyard, he must walk the length 25 times or walk its perimeter 10 times. What is the area of Maartin's backyard in square meters?

46. **(AJHSME, 2000)** A cube has edge length 2. Suppose that we glue a cube of edge length 1 on top of the big cube so that one of its faces rests entirely on the top face of the larger cube. Find the percent increase in the surface area (sides, top and bottom) from the original cube to the new object made from the two cubes.

47. **(AJHSME, 2000)** In the figure on the below $\angle A = 20°$ and $\angle AFG = \angle AGF$. Find the size of $\angle B + \angle D$.

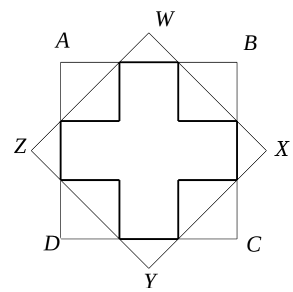


48. **(AJHSME, 2000)** The area of the rectangle $ABCD$ below is 72. If point A and the midpoints of BC and CD are connected to form a triangle, what is the area of that triangle?

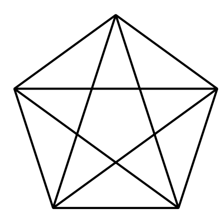


2 Intermediate Problems

1. In the figure below a cross shape with 12 edges (the darker segments), each of length 1, is inscribed in two squres $ABCD$ and $WXYZ$. Find the area of each of these two squares.



2. The figure below shows a regular pentagon, an inscribed pentagram and the segments of the pentagram enclose a smaller pentagon. The perimeter of the inner pentagon is 5 cm and the perimeter of the pentagram is $10x$ cm. Show that the perimeter of the outer pentagon is $5x^2$ cm.



3. The rectangle $ABCD$ has $AB = 15$ and $AD = 10$. P is the point inside the rectangle for which $AP = 10$ and $DP = 12$. Find the angle DPC.

4. Triangle $\triangle ABC$ has $AB = AC = 5$ and $BC = 6$. From any point P, inside or on the boundary of this triangle, line segments are drawn at right angles to the sides; the lengths of these segments are x, y and z.

   (a) Find the largest possible value of the total $x + y + z$ and find the positions of P where this largest total occurs.

   (b) Find the smallest value of the total $x + y + z$ and find the positions of P where this smallest total occurs.

   (c) What if $\triangle ABC$ is a general (non-isosceles) triangle?

5. In triangle $\triangle ABC$, $\angle ACB$ is a right angle, $BC = 12$ and D is a point on AC such that $AD = 7$ and $DC = 9$. The perpendicular from D to AB meets AB at P and the perpendicular from C to BD meets BD at Q. Calculate:

   (a) The ratio of the area of triangle $\triangle BCD$ to the area of triangle $\triangle BAD$.

   (b) The ratio of the length of QC to the length of PD.

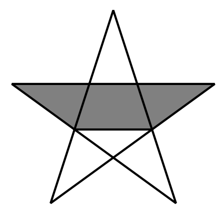
6. A square has one corner folded oer to create a pentagon. The three shorter sides of the pentagon which is formed are all the same length. Find the area of the pentagon a a fraction of the area of the original square.

7. A square is inscribed inside a quadrant of a circle of radius 1. Calculate the area of the square.

8. In a triangle the length of one side is 3.8 and the length of another side is 0.6. Find the length of the third side if it is known that it is an integer.

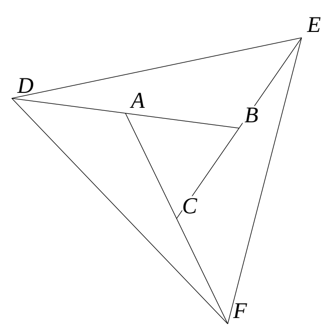
9. Using a pen and a straight edge, draw on a square grid a square whose area is:
   (a) twice the area of one square of the grid.
   (b) 5 times the area of one square of the grid.

10. In the figure below find the area of the shaded region as a fraction of the area of the entire regular pentagram.



11. Is it true that a laf liter bottle of Coke is proportional to a liter bottle of Coke, i.e., one can be obtained from another by multiplying all the lengths by the same factor?

12. Suppose that triangle $\triangle ABC$ has an area of 1. Plot points D, E and F so that A is the midpoint of BD, B is the midpoint of EC and C is the midpoint of AF. What is the area of $\triangle DEF$?



3 Harder Problems

In some of the questions that follow we will need to recall the meaning of the term "convex". A convex planar figure is the intersection of a number (finite or infinite) of half-planes. The intersection of a finite number of half-planes is a convex polygon. (Equivalently, a figure is convex if for any two points A and B of the figure, the entier line segment AB belongs to the figure.

1. Suppose that $ABCD$ is a convex quadrilateral. Extend its sides AB, BC, CD and DA so that B is the midpoint of AB_1, C is the midpoint of BC_1, D is the midpoint of CD_1 and A is the midpoint of DA_1. If the area of the quadrilateral $ABCD$ is 1, find the area of $A_1B_1C_1D_1$.

2. Let $ABCD$ be a trapezoid with $AD \| BC$ and let O be the point of intersection of the diagonals AC and BD. Prove that the triangles $\triangle AOB$ and $\triangle DOC$ have the same area.

3. Let $\triangle ABC$ be a triangle and suppose that P is an arbitrary point of AB. Find a line through P that divides $\triangle ABC$ into two regions of equal area.

4. Let $ABCD$ be a convex quadrilateral. Find a line through the vertex A that divides $ABCD$ into two regions of equal area.

5. Two lines trisect each of two opposite sides of a convex quadrilateral. Prove that the area of the part of the quadrilateral contained between the lines is one third of the area of the quadrilateral.

6. Suppose that $ABCD$ is a convex quadrilateral. Two points are given on each of the four sides of this quadrilateral that trisect the sides: K and M on AB, P and R on BC, N and L on CD and S and Q on DA. By the trisection we have: $AK = KM = MB$, $BP = PR = RC$, $CN = NL = LD$ and $DS = SQ = QA$. Let A_1 be the intersection of KL and PQ, B_1 is the intersection of MN and PQ, $C_!$ is the intersection of MN and RS and D_1 is the intersection of KL and RS. Prove that the area of $A_1B_1C_1D_1$ is $1/9$ of that of $ABCD$.

7. In a certain country, there are 100 airports and all the distances between them are different. An airplane takes off from each airport and lands at the closest airport. Prove that none of the 100 airports receives more than 5 planes.

8. n points are placed in a plane in such a way that the area of every triangle with vertices at any three of these points is at most 1. Prove that all these points can be covered by a triangle with an area of 4.

9. Is it possible to place 1000 line segments in a plane in such a way that every endpoint of each of these line segments is at the same time an inner point of another segment?

10. Suppose that points A, B, C and D are coplanar but no three are collinear. Prove that at least one of the triangles formed by these points is not acute.

11. In a coordinate plane there are infinitely many rectangles. Vertices of every rectangle have coordinates $(0,0)$, $(0,m)$, $(n,0)$ and (n,m), where m and n are positive integers (different for different rectangles). Prove that it is possible to choose two of these rectangles so that one is completely covered by the other.

12. Prove that

    (a) any convex polygon of area 1 can be covered by a parallelogram of area 2.

    (b) a triangle of area 1 cannot be covered by a parallelogram of area less than 2.

13. (a) Suppose that there are four convex figures in a plane, and every three of them have a common point. Prove that all four figures have a common point.

    (b) Suppose that there are n convex figures in a plane, and every three of them have a common point. Is it necessarily true that all n figures have a common point?

14. A number of line segments lie in a plane in such a way that for any three of them there exists a line intersecting them. Prove that there exists a line intersecting all these segments.

15. Is it true that for every pentagon it is possible to find at least two sides such that the pentagon belongs to the intersection of exactly two half-planes determined by these sides?

16. Draw a polygon and a point P inside this polygon so that none of the sides is completely visible from the point P.

17. Draw a polygon and a point P outside of this polygon such that none of the sides is completely visible from the point P.

18. If a rectangle can be covered completely with 100 circles of radius 2, show that it can also be covered by 400 circles of radius 1.

19. (a) Prove that every n-gon (with $n \geq 4$) has at least one diagonal that is completely contained inside the n-gon.

    (b) Find the least possible number of such diagonals.

    (c) Prove that every polygon can be cut into triangles by non-intersecting diagonals.

    (d) Suppose that a polygon is cut into triangles by non-intersecting diagonals. Prove that it is possible to color the vertices of the polygon using three colors in such a way that all three vertices of each of the triangles are different.

20. Is it possible to cover a 10×10 "chessboard" by 25 1×4 dominoes?

4 Solutions: Easier Problems

1. Solution: 34. Ignoring the sides that are listed as having lengths 10 and 7, although the size of the corner in the lower-right corner is unknown, the total horizontal part of the remaining sides is 10 and the total vertical part measures 7.

2. For the figure on the left, if we subtract the areas of the two smaller circles from the area of the large circle, the remaining shaded and unshaded parts of the original circle must have equal areas. Thus the shaded part has half the area of the remaining part, so it will be:

$$\frac{\pi \cdot 2^2 - 2\pi \cdot 1^2}{2} = \pi.$$

   The perimeter will be half of the total perimeter of the three circles, or:

$$\frac{4\pi + 2 \cdot 2\pi}{2} = 3\pi.$$

   For the figure on the right (with three circles) the same arguments can be made. The area will be smaller, but perhaps surprisingly, the perimeter will be the same. To show that, and to answer the final questions about the same situation with n circles, we'll work out the result for arbitrary n and we can then simply substitute $n = 3$ to find the corresponding areas and perimeters in that special case. (**Note:** We can also check our algebra by substituting $n = 2$ and make sure that we obtain the same results we did for the case with two circles.)

   With n circles inside a large circle of radius 1, each will have radius $1/n$. The area of each small circle will thus be π/n^2. However, there are n of them, so the total area to be subtracted is $n \cdot (\pi/n^2) = \pi/n$. The area of the larger circle is $\pi \cdot 1^2 = \pi$, so the area of the shaded part will be half of $\pi - \pi/n$ which is $\frac{(n-1)\pi}{2n} = \frac{n-1}{2n}\pi$. As n gets large, the fraction $\frac{n-1}{2n}$ gets very close to $1/2$, so the area of the shaded region gets close to $\pi/2$, which is half the area of the large circle, which sort of seems reasonable.

   The perimeters, however, do something different. A circle of radius $1/n$ has perimeter $\frac{2\pi}{n}$, and there are n of them, so the total perimeter of the n small circles is 2π. The perimeter of the larger circle is also 2π, so the perimeter of the shaded region would be half of $2\pi + 2\pi$, or 2π—it doesn't depend on n at all.

3. There is not enough information to calculate the area. You can see why by assigning variables to the two diameters (which need to add to 2) and when you work out the area, you can see that it depends on the diameters. But it is also obvious if you simply imagine that one of the two circles is tiny so the other is almost as big as the surrounding circle, making the shaded area almost zero. However, if you work out the perimeter of that shaded area, you'll find that it is, again, the same as in the previous problem, independent of the relative sizes of the two circles.

4. Solution: $1/2$. For each of the shaded areas, there is an equal-sized non-shaded area on the other side of the diagonal line. That means that the shaded and non-shaded areas are the same. The total area is 1 and it's made of two equal areas, so each of those areas is $1/24$.

5. If the total area is 100 and the squares are equal, each has area $100/4 = 25$. That means that each of the small squares is 5×5, so each has an edge-length of 5. Eight of these edges make the perimeter, so the perimeter is $8 \cdot 5 = 40$.

   If the perimeter is 100, it is made of eight equal-length square edges, so each edge has length $100/8 = 25/2$. Thus the area of each square is $(25/2)^2 = 625/4$. But the total area is made of four equal squares, giving a total area of 625.

   A perhaps easier way to find the area in the second case once you've worked out the first part of the problem is to note that the area of similar figures varies as the square of the length. The new perimeter is 100 and the old one is 40 so the new area must be $(100/40)^2 = (5/2)^2 = 6.25$ times the old area. The old area is 100, so the new one must be $100 \cdot 6.25 = 625$.

6. Solution: 36. Each time you cut off a corner, three edges are added (and none disappear). There are 8 corners and 12 edges originally, so the answer is $12 + 3 \cdot 8 = 36$.

   Note that the problem is actually ill-formed. If large chunks are taken out of each corner, then the cuts might overlap or touch, so almost any shape with 8 or fewer faces could be formed, with a large variety of possible answers.

7. Solution: $50°$. By the exterior angle theorem, $\angle B = \angle A + \angle E$. (If you don't know the exterior angle theorem, note that $\angle B$ plus its supplement is $180°$ and since the three angles of a triangle add to $180°$ we can conclude the same thing. We also know that $\angle B + \angle C + \angle BDC = 180°$, so a little arithmetic gives us $\angle BDC = 50°$.

8. Solution: all three shaded areas are the same; namely, $1/4$ of the area of the square. One way to see this is to divide each figure up into a bunch of identically-shaped triangles or squares (which has already been done in figure II) and count the number that are shaded. In figure I, for example, divide the square horizontally and along the other diagonal to make 8 congruent triangles, two of which are shaded, making an area of $2/8 = 1/4$. Figure III can be divided by the horizontal and vertical bisectors of the square to from 16 identical triangles, 4 of which are shaded, making the area $4/16 = 1/4$.

   It's also easy to see that the area of figure I is the same as that of figure II by sliding the lower triangle in figure I up and to the right until it's next to the other shaded part, and forms a figure that's identical (with a rotation) to figure II.

9. Solution: $9 : 1$. If a perimeter is three times another, that means that all the sides are three times as long. If the side of the smaller square has length L its area will be L^2. For a square with side length $3L$, the area will be $(3L)^2 = 9L^2$, so the ratio of the areas will be $9 : 1$.

   The question on surface area is more interesting. If the smaller square has six faces each with area A, then the cube with surface area 3 times that will have six square faces, each with area $3A$. The length of the edge of the smaller cube will thus be $\sqrt{A}$ and of the larger cube, $\sqrt{3A}$. To obtain the volumes, we need to cube the lengths of the sides. The smaller square's volume will be $A^{3/2}$ and of the larger one, $(3A)^{3/2} = 3^{3/2}A^{3/2}$. The ratio will be $3^{3/2} = 3\sqrt{3}$.

10. Solution: 12. The radius of each semicircle is half the length of the side of the inner square, so the outer square, $ABCD$, has sides that are twice as long. If you double the length of the side, you multiply the area by 4, so we obtain 12.

To show this with algebra, if the side of the smaller square is L, the area is L^2. Doubling L to $2L$ makes the area $(2L)^2 = 4L^2$, or 4 times the smaller area.

11. Solution: $9/2 + 3\sqrt{2}$. To find the area, we need to find the length of the sides of the larger square. Draw a line straight across it from where two of the semicircles touch the outer square. That line will pass through the midpoints of two adjacent sides of the inner square. If the inner square has side length L, then the paths of the line you drew through the semicircles will be two times the radius, for a total of L. The diagonal part through the inner square will have length $L/\sqrt{2}$, so the total length of the line will be $L + L/\sqrt{2} = L(1 + \sqrt{2}/2)$. The ratio of the side lengths of the two squares is $(1 + \sqrt{2}/2)$, so the ratio of the areas is the square of that, and since the smaller square has area 3, the area of the larger square is:

$$3 \cdot \left(1 + \frac{\sqrt{2}}{2}\right)^2 = 9/2 + 3\sqrt{2}.$$

12. To find the side length of a square, divide the perimeter by 4 (since the four sides are equal-length). So the side length of square I is 2.5 and of II is 5. From the diagram, the side length of square III must be $2.5 + 5 = 7.5$. From this data, we can find that the perimeter of III is $4 \cdot 7.5 = 30$ and the perimeter of the combined figure is 40.

13. Solution: 32. The height of the rectangle is the diameter of the circles, and the width is twice the diameter of the circles, so the area is $4 \cdot 2 \cdot 4 = 32$.

14. Solution: $85°$. Triangle $\triangle BED$ is an isosceles right triangle, so its other two angles are both $45°$. So $\angle AED = 45° + 40° = 95°$. Since the four angles of a convex quadrilateral add to $360°$ and we know three of them: $90°$, $90°$ and $95°$ we can find that the fourth must be $85°$.

15. Solution: 50×50 inches. Since there are 4 identical L-shaped regions, their total area must be $12/16 = 3/4$ of the large square's area, or 7500 square inches. That leaves 2500 square inches for the inner square, so its side length must be $\sqrt{2500} = 50$.

    Another way to see it is that once you know the area of the inner square is $1/4$ that of the entire square, its corresponding lengths much be $\sqrt{1/4} = 1/2$ of the larger one and half of 100 is 50.

16. Solution: 7.2. The area of the parallelogram is the altitude $DE = 6$ times the base $AB = DC = 12$, or $6 \cdot 12 = 72$. Looking at the altitude from a different direction, DF is an altitude if the base is $CB = AD$. Since $EB = 4$, AE must be 8 since the sum is 12. (Obviously, the picture is not drawn to scale.) But $\triangle DEA$ is a right triangle with sides equal to 6 and 8. By the Pythagorean theorem, The length of DA is $\sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$, so $10 \cdot DF = 72$.

17. Solution: $1/2$. There are a bunch of ways to do this:

    (a) The result is instant using Pick's theorem, which states that for any simple lattice polygon, the area is given by the formula:

    $$\mathcal{A} = I + B/2 - 1,$$

    where I is the number of lattice points interior to the polygon and B is the number on the boundary. In this case, $I = 0$ and $B = 3$, yielding an area of $1/2$.

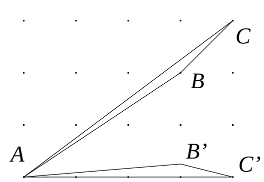
(b) A more straightforward method, perhaps the most obvious, is to take the area of the entire square and to subtract off the areas of triangles and rectangles that are not part of $\triangle ABC$. The total area is 12, the rest of the non-$\triangle ABC$ can be split easily into three triangles and one rectangle having areas 6, 3, $1/2$ (for the triangles) and 2 (for the rectangle formed by drawing lines down and to the right from point B. Thus the area is $12 - 6 - 3 - 1/2 - 2 = 1/2$.

(c) Another very nice method is to note that if BC is considered to be the base, then any point on the line through A and parallel to BC would form a triangle with the same area, since it has the same height. One such point (call it A') is the one immediately to the left of C in the diagram and it's obvious that the area of $\triangle A'BC$ is $1/2$.

(d) Cavalieri's principle states that:

Suppose two regions in a plane are included between two parallel lines in that plane. If every line parallel to these two lines intersects both regions in line segments of equal length, then the two regions have equal areas.

Using this rule, we can simply project our triangle in a suitable direction to make another whose area is easier to compute. For our example, let's project $\triangle ABC$ down onto $\triangle AB'C'$ as pictured below:



It's easy to calculate the height of B' over the line AC' and that is $1/4$. The area of $\triangle AB'C'$ is thus $1/2$.

(e) Another easy solution uses the following formula for the area of a simple n-sided polygon where the coordinates of the vertices are known:

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}\Big[(x_0y_1 - y_0x_1) + (x_1y_2 - y_1x_2) + \\ &\quad \cdots + (x_{n-2}y_{n-1} - y_{n-2}x_{n-1}) + (x_{n-1}y_0 - y_{n-1}x_0)\Big],\end{aligned}$$

where the coordinates of the points, in counter-clockwise order, are given by: (x_0, y_0), (x_1, y_1), $\ldots$, (x_{n-1}, y_{n-1}).

We need to assign coordinates, so let's let the coordinates of the lower-left vertex be $(x_0, y_0) = (0, 0)$ making the others $(x_1, y_1) = (3, 2)$ and $(x_2, y_2) = (4, 3)$. The formula yields:

$$\mathcal{A} = \frac{1}{2}\Big[(0 \cdot 2 - 3 \cdot 0) + (3 \cdot 3 - 2 \cdot 4) + (4 \cdot 0 - 0 \cdot 3)\Big] = \frac{1}{2}.$$

(f) We can also calculate the lengths of the sides using the Pythagorean theorem and then apply Heron's formula:

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b and c are the lengths of the sides of the triangle and $s = (a + b + c)/2$ is the semiperimeter.

In our case, the Pythagorean theorem yields the three lengths: $AB = c = \sqrt{13}$, $BC = a = \sqrt{2}$ and $AC = b = \sqrt{25} = 5$, so $s = (\sqrt{13}+\sqrt{2}+5)/2$, so the area is given by:

$$\mathcal{A} = \sqrt{\begin{array}{c}\frac{1}{2}\left(5+\sqrt{2}+\sqrt{13}\right)\cdot\left(\frac{1}{2}\left(-5+\sqrt{2}+\sqrt{13}\right)\right)\cdot\\ \left(\frac{1}{2}\left(5-\sqrt{2}+\sqrt{13}\right)\right)\cdot\left(\frac{1}{2}\left(5+\sqrt{2}-\sqrt{13}\right)\right)\end{array}},$$

which, with some really ugly algebra that's required to expand the product above, yields $\mathcal{A} = 1/2$.

(g) If we know one of the angles and the lengths of the two adjacent sides, we can also find the area. In the figure, let the lengths of the sides opposite vertices A, B and C be a, b and c as in the previous solution. The area is given by:

$$\mathcal{A} = (ab\sin C)/2.$$

To find angle C we use the law of cosines:

$$\cos C = (a^2+b^2-c^2)/(2ab) = (2+25-13)/(10\sqrt{2}) = 7\sqrt{2}/10.$$

We know that $\sin C = \sqrt{1-\cos^2 C} = \sqrt{1/50}$, so the area is given by:

$$\mathcal{A} = (5\sqrt{2}\sqrt{1/50})/2 = 1/2.$$

(h) The cross product of two (3-D) vectors is a vector perpendicular to both and whose length is the area of the parallelogram determined by the two vectors. So if we add a z-component of 0 to each of two of our vectors and take the cross product, the only component we need to calculate is the z component of that, and divide by 2. If we take the two vectors in the wrong order we'll get the wrong answer but it will be the same size, only negative, so we can take the absolute value of that for the parallelogram's area. Then divide by 2 for the area of the triangle.
So let's do this, starting from point B. $\overrightarrow{BC} = (1,1)$ and $\overrightarrow{BA} = (-3,-2)$. Take the cross-product:

$$(1,1,0)\times(-3,-2,0) = (0,0,1\cdot(-2)-(-3)\cdot 1) = (0,0,1).$$

Thus the area is (the absolute value of) half the z-coordinate of the cross product, or $1/2$.

(i) Set up a coordinate system with point A at the origin and note that the equations of the lines are:

$$\begin{array}{rcl} AC & : & y = f(x) = 3x/4 \\ AB & : & y = g(x) = 2x/3 \\ BC & : & y = h(x) = x-1 \end{array}$$

The area is specified by the following difference of definite integrals:

$$\mathcal{A} = \int_0^4 f(x)\,dx - \int_0^3 g(x)\,dx - \int_3^4 h(x)\,dx.$$

Using standard integration techniques, we obtain:

$$\mathcal{A} = \left[\frac{3x^2}{8}\right]_0^4 - \left[\frac{x^2}{3}\right]_0^3 - \left[\frac{x^2}{2}-x\right]_3^4 = 6-0-(3-0)-(4-3/2) = 1/2.$$

(j) If you like to do line integrals, then you can apply Green's Theorem (a corollary to which) states that if C is a counter-clockwise path around the boundary of an area, such that no vertical or horizontal line intersects the boundary in more than 2 points, then the area enclosed by the curve C is given by:

$$\mathcal{A} = \oint_C x\,dy - y\,dx.$$

This calculation is very similar to the one above.

There are, no doubt, other ways to calculate the area as well.

18. Solution: 3. This occurs when A, B, C and D are all on the same line in the order $ADCB$.

19. The area of the square $OPQR$ is 4, so to make a triangle with height 2 have the same area, its base must have length 4. Thus the coordinates of T must be $(-2, 0)$.

    The second question is trickier. Suppose T is x distance to the left of the origin. Let y be the length of the segment SO. Then since triangles $\triangle SOT$ and $\triangle QRT$ are similar, we have: $y/x = 2/(2+x)$, so $y = 2x/(2+x)$. The area of triangle $\triangle SOT$ is thus: $xy/2 = x^2/(2+x)$. Set this equal to 4 and we obtain the quadratic equation:

    $$x^2 - 4x - 8 = 0.$$

    If we solve for x, perhaps using the quadratic formula, we obtain $x = 4(1+\sqrt{3})$, so the coordinates of T are $(-4(1+\sqrt{3}), 0)$.

20. Since the three angles of a triangle add to $180°$, we know that $\angle BAC + \angle BCA = 130°$. If we divide that equation by 2 on both sides we find that $\angle DAC + \angle DCA = 65°$. So $\angle ADC = 115°$ since the three angles in $\triangle ADC$ also add to $180°$.

21. A square surrounding the circle of radius 4 must have a side of length 8 (the diameter of the circle). Thus the area is $8 \cdot 8 = 64$. The diagonal of the largest square that will fit inside such a circle is also 8. Using the Pythagorean theorem we can conclude that the side length of such a square is $8/\sqrt{2}$ and the area of that square will be 32.d

22. If the stripe width is 1 then the side length of the square is 6 so its area is $6^2 = 36$. The areas of the shaded V-shaped areas are $3, 7$ and 11, so the required fraction is $21/36 = 7/12$.

23. Solution: $35°$. Since we know $\angle ABC = 70°$ and $\angle BAC = 40°$ we can conclude that $\angle ACB = 70°$ since the three add to $180°$. Since it is supplementary to $\angle ACB$, $\angle ECD = 110°$. $\angle CDE$, $\angle CED$ and $110°$ also add to $180°$ and since the other two angles are equal, each, including $\angle CED$, is equal to $35°$.

24. Solution: $5:9$. The area ratios will be the same no matter how big the squares, so assume that the large square has side length equal to 3. Then the side length of the inner square is the hypotenuse of a right triangle with sides 1 and 2, so its length is $\sqrt{1^2+2^2} = \sqrt{5}$ by the Pythagorean theorem. The area of the larger square is $3^2 = 9$ and that of the inner square: $(\sqrt{5})^2 = 5$, so the required ratio is $5:9$.

25. Solution: 16. Each face has 2 diagonals for a total of 12 and there is only one interior diagonal from each vertex. There are 8 vertices, but each interior diagonal uses 2 of them so there are 4 additional interior diagonals, making a total of 16.

26. The original surface area of the cube is $9 \cdot 6 = 54$ cm^2. Removing a corner cube subtracts three faces, but adds three interior ones, so the surface area remains unchanged. Doing the same for all eight corners does the same: has no effect on the original cube's surface area. On the other hand, when a face cube is removed, five faces are introduced and one is removed for a net gain of four. Since there are six faces, removing all the center face cubes will add $6 \cdot 4 = 24$ cm^2 for a total surface area of $54 + 24 = 78$ cm^2.

27. Solution: 3 : 2. The ratio of the areas will be independent of the original circle size, so we choose a convenient radius for the outer circle; namely, 5. That will make the radius of the semicircle ABC equal to 2 and of the semicircle CDE, 3.

    Think of the area of the upper shaded region as a semicircle of the large circle where we subtract off the area of the semicircle ABC and add the area of the semicircle CDE. A similar result can be obtained for the lower region, or simply subtract the area of the upper from the area of the entire circle. The area of the large circle is 25π so the area of the large semicircle is $25\pi/2$. The area of semicircle ABC is $4\pi/2$ and of the semicircle CDE, $9\pi/2$. The area of the upper (shaded) region is thus:

    $$\frac{25\pi}{2} - \frac{4\pi}{2} + \frac{9\pi}{2} = \frac{30\pi}{2}.$$

    The area of the lower region is similarly:

    $$\frac{25\pi}{2} + \frac{4\pi}{2} - \frac{9\pi}{2} = \frac{20\pi}{2}.$$

    The desired ratio is thus 3 : 2.

28. What makes this problem difficult is that it's necessary to count triangles of every size. In the figure on the left, there are 5. On the right, there are still the same 5 from the other figure, but the new lines add 8 more for a total of 13. One possible way to reduce errors is to add the two new lines one at a time and see how many new triangles are formed with the additions.

29. Solution: 6. The easiest way to see this is that there is basically a rectangle of area 6 but with the addition and subtraction of identically-shaped triangular regions.

30. Solution: 54. Either use the same reasoning as in the figure above, or note that the lower figure is exactly the same as the upper one, but with width and height multiplied by 3, so there will be $3^2 = 9$ times the area of the figure in the previous problem.

31. Solution: 1 : 8. If the larger square has side length 1, the smaller square has side length $1/4$ of the diagonal, or $\sqrt{2}/4$. The areas are the squares of these two numbers; namely, 1 and $2/16 = 1/8$, yielding a ratio of 1 : 8.

32. Solution: 24. Each fold cuts the area in half, so the original square's area must have been 36. Its side length was therefore $\sqrt{36} = 6$ so its perimeter is 24.

33. Solution: 52. The easy way to do this is to notice that there are a total of $4 \cdot 4 \cdot 4 = 64$ small cubes. Of those, there's a block of $2 \cdot 2 \cdot 3 = 12$ of them that don't touch a side or bottom, so the answer is $64 - 12 = 52$.

    A harder way to do it is to count the cubes on the sides and bottom directly, but this is a bit tricky. There are 5 of the 2×2 cubes on the faces that are not on edges or corners for a total of 20. There are 12 edges that contain 2 cubes for a total of 24 more. Finally, there are 8 corners with one cube in each for a grand total of $20 + 24 + 8 = 52$ cubes.

34. Solution: $60°$. The difference between the two hands will be $2/12 = 1/6$ of a circle, and a full circle is $360°$, so the angle is $(1/6) \cdot 360° = 60°$.

35. Solution: 400 foot^2. The original area is $10 \cdot 50 = 500$, and the perimeter of the fence is 120. If the fence is arranged in a square, each side will be 30, so the area of the larger garden will be $30^2 = 900$. Thus the area is increased by 400 foot^2.

36. Solution: Blue.

37. Solution: 34. The segment AD is 8 longer than BC, so if we form a right triangle by dropping a perpendicular from B to AD it will form a right triangle with sides 3 and 4. By the Pythagorean theorem, the length of AB will be $\sqrt{3^2 + 4^2} = 5$, so the perimeter of $ABCD$ is 34.

38. Solution: $30°$. We use the fact that the angles of a triangle add to $180°$ twice. In $\triangle BED$ we can conclude that $\angle D = 40°$. Next, looking at $\triangle ACD$ we find that $\angle A = 30°$.

39. Solution: $\sqrt{13}$. By the symmetry of the figure, if we add the line segment CA then $\triangle CAN$ will have half the area of $\triangle CND$ so the base AN must be half of the base ND. Thus $AM = 1$ and $BM = 2$ so by the Pythagorean theorem, $CM = \sqrt{2^2 + 3^2} = \sqrt{13}$.

40. Solution: 8.00. If we take slides at DF, GI, and so on, we note that the shaded area in each slice is $1/3$ of the total area. Thus with 100 such slices, we will be very close to $1/3$ of the area of the entire triangle, which is $(1/3) \cdot 24 = 8$.

41. Solution: 7. The width of the strip is 1, so its area is 7.

42. Solution: 353. We need 50 blocks to go 100 feet if they all are the longer blocks, and 51 of them if there are shorter blocks at the end. We need $4 \cdot 50 + 3 \cdot 51 = 353$ blocks.

43. Solution: $72°$. Since we know angles C and A, then the large angle at B is $72°$. Bisecting it yields $\angle CTR = 36°$ and since the three angles of $\triangle CRT$ add to $180°$ we know that $\angle CRT = 72°$.

44. Solution: 15. $AB = 4$ so $AD = 2$ and $EF = 1$ (as well as the others of similar lengths. The perimeter of the complex figure is the sum of the lengths of AB, BC, CD, DE, EF, FG, and GA or $4 + 4 + 2 + 2 + 1 + 1 + 1 = 15$.

45. Solution: 400. The length is $1000/25 = 40$ and the perimeter is $1000/10 = 100$. Thus the perimeter has two sides of length 40 and two of length 10. The area is $40 \cdot 10 = 400$ meter^2.

46. Solution: $16\ 2/3\%$. The surface area of the original cube was $4 \cdot 6 = 24$. The surface area of the smaller cube is 6, but when it is glued to the larger cube, two faces of area 1 are eliminated, so the new object has area $24 + 6 - 2 = 28$. The surface area increase is $4/24 = 1/6 = 16\ 2/3\%$.

47. Solution: $80°$. We know that $\triangle AFG$ is isosceles, so $\angle AFG$ must be $80°$. That means that $\angle BFD = 100°$. Since the three angles of a triangle add to $180°$, $\angle B + \angle D = 80°$.

48. Solution: 27. If we label the midpoint of CD as X and the midpoint of BC as Y we know that the areas of $\triangle ABY$ and $\triangle ADX$ are $1/4$ the area of the rectangle, and that the area of $\triangle XYC$ is $1/8$ the area. Thus the area of $\triangle AXY$ is $1 - 1/4 - 1/4 - 1/8 = 3/8$ the area of the rectangle, or 27.

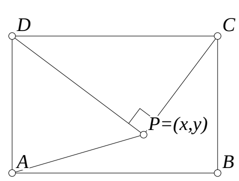
5 Solutions: Intermediate Problems

1. The area of the inner square $ABCD$ is easy: the lengths of its sides is 3 so the area is 9. The the sides of $WXYZ$ are made up of sides or hypotenuses of isosceles right triangles. The outer (smaller) triangles have sides of length $1/\sqrt{2} = \sqrt{2}/2$ and there are two of these along each side, for a length of $\sqrt{2}$. The hypotenuse that lies inside the square $ABCD$ has length $\sqrt{2}$ so the total side length of square $WXYZ$ is $2\sqrt{2}$ so its area is 8.

   Another nice way to obtain the area of 8 is to note that square $WXYZ$ is formed of 5 1×1 squares, 4 half-squares, and 4 quarter-squares, making a total of 8 full squares.

2. The pentagram's perimeter is made of 10 equal-length segments, so each of those must have length x. The isosceles triangles formed with a side of the inner pentagon (and thus having length 1) is a $36° - 72° - 72°$ triangle and hence the longer sides have length τ, the golden ratio. ($\tau = (1 + \sqrt{5})/2$.) The outer isosceles triangles having the shorter sides of length τ are $144° - 36° - 36°$ triangles and so the longer edge is τ times the shorter lengths, and hence is $\tau^2 = x^2$. Thus the perimeter of the outer pentagon is $5x^2$.

3. See the figure below. Assign coordinates so that $A = (0, 0)$, $D = (0, 10)$, $B = (15, 0)$ and $P = (x, y)$.



   Since $AP = 10$ and $DP = 12$ we have the following equations:

$$\begin{aligned} x^2 + y^2 &= 10^2 \\ x^2 + (y - 10)^2 &= 12^2 \end{aligned}$$

Subtract the two equations to obtain $20y = 56$, or $y = 14/5$ and when we plug this back into one of the equations to solve for x we obtain $x = 48/5$.

Calculate the length CP as follows:

$$\begin{aligned} CP &= \sqrt{\left(10 - \frac{14}{5}\right)^2 + \left(15 - \frac{48}{5}\right)^2} \\ &= \sqrt{\left(\frac{36}{5}\right)^2 + \left(\frac{27}{5}\right)^2} \\ &= \sqrt{\frac{2025}{25}} = \frac{45}{5} = 9. \end{aligned}$$

Now $\triangle CPD$ has sides $9, 12$ and 15 so it is a right triangle and thus $\angle DPC = 90°$.

4. Imagine line segments drawn from P to the vertices of the triangle, dividing it into three smaller triangles. The sum of the areas of these three triangles is constant, as long as P is inside the larger triangle. But the sum of the areas is just $1/2$ times the sum of the perpendicular lengths x, y and z to the sides of the larger triangle, something like:

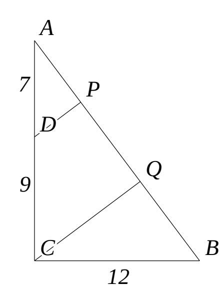
$$\frac{5x + 5y + 6z}{2},$$

which is a constant (the area of $\triangle ABC$). To make the sum as large as possible, z needs to be as small as possible, so set z to zero and the sum $x+y+z = x+y$. Now $5x + 5y$ is constant (twice the area of the large triangle, and this will occur whenever z is on the line of length 6 (line BC).

To make the sum as small as possible, set $x = y = 0$, so z is at the vertex A.

For a general triangle, the same idea holds, but the point P needs to be either at the vertex opposite the shortest edge (for the largest sum) or at the vertex opposite the longest edge (for the smallest sum).

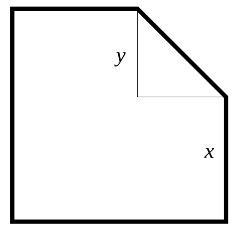
5. See the figure below.



If we consider AC to be the base, then triangles $\triangle BCD$ and $\triangle BAD$ have the same height, so the ratio of their areas is the same as the ratio of their bases; namely, 9 : 7.

We also know that $\triangle APD \sim \triangle AQC$ so all their side lengths are proportional. Thus $QC : PD = 16 : 7$.

6. Assume the square has area 1 (so is 1×1). In the figure below we need to find x and y such that $x + y = 1$ and so that the folded part is equal to x in length.



The folded part has length $x = y\sqrt{2}$, so we have:

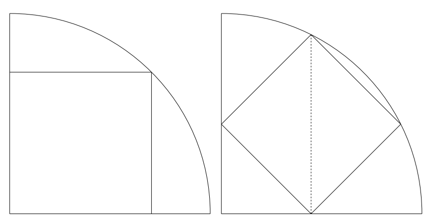
$$y + y\sqrt{2} = 1.$$

Solve to y to obtain $y = \sqrt{2} - 1$.

The area of the pentagon is the area of the square (1) minus the area of the triangle that is cut off. So the pentagon's area is:

$$1 - (\sqrt{2} - 1)^2/2 = \sqrt{2} - 1/2.$$

7. There are two ways to inscribe a square in a quarter circle; see below:

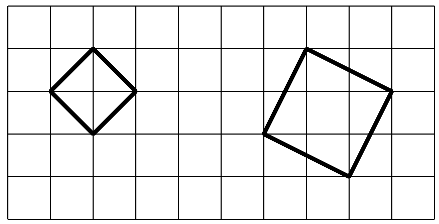


On the left, it's clear that the diagonal of the square is the same as the radius of the circle, so if the diagonal is 1, by the Pythagorean theorem, the length of the side is $\sqrt{2}/2$, so the area is $1/2$.

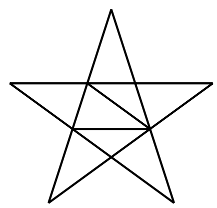
If the square is inscribed as in the figure on the right, it's a little trickier, but consider the diagonal of the square. If we connect the top of the diagonal to the corner of the square it forms a right triangle with hypotenuse 1 and with width the same as half the height. If the width is w then the height is $2w$ so the Pythagorean theorem gives: $w^2 + (2w)^2 = 1^2$, so $w = 1/\sqrt{5}$, so the full diagonal of the square is $2/\sqrt{5}$, making the side of the square equal to $2/\sqrt{10}$. Square that to obtain the square's area and we obtain: $4/10 = 2/5$.

8. If the side of length 0.6 pivots one way it can form a line as long as 4.4; the other way, as short as 3.2. The length of the third side must be between 3.2 and 4.4. Since its length is an integer, it must be 4.

9. In the figure below, using the Pythagorean theorem, it's easy to see that the side length of one of the squares is $\sqrt{2}$ and that of the other, $\sqrt{5}$, so the two squares have areas 2 and 5..



10. Consider the figure below which divides the pentagram into two sorts of regions: 6 of one type and 2 of the other.



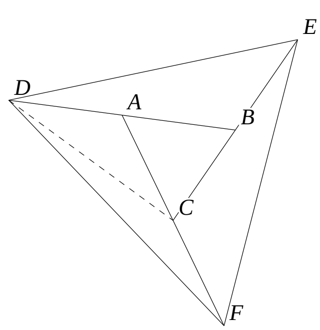
Because of the geometry of a pentagram, there are only two side lengths of the triangles in the figure. If the shorter one has length 1 then the other has length $\tau = (1+\sqrt{5})/2$. (τ is the golden ratio.)

Consider one of the triangles formed with one of the smaller triangles and one of the larger ones. The ratio of their areas will be the same as the ratio of their base lengths, since the heights are the same. Thus the ratios of the areas of the two types of triangles are $1 : \tau$.

If we call the small triangle's area 1, then the area of the pentagram will be: $2 + 6\tau$. If we calculate the area of the shaded region in the original problem, we obtain $1 + 3\tau$. Thus the shaded region has half the area of the full pentagram.

11. If you look at soft drink bottles carefully, you'll note that the caps are all the same size, so the bottles are not exactly similar: some parts are larger and some parts are the same size.

12. Construct the line DC as in the figure below:



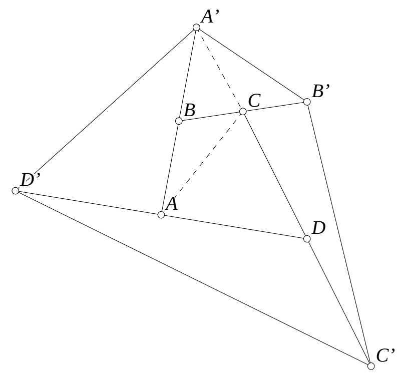
Since C is the midpoint of AB then the areas of triangles $\triangle ACD$ and $\triangle CFD$ are the same (they have equal bases). Similarly, since A is the midpoint of DB

we know that triangles $\triangle ABC$ and $\triangle ADC$ are the same for the same reason. Thus all three triangles have the same area.

We can do the same thing by constructing lines AE and BF that, using similar arguments, will divide the triangle DEF into 7 regions, each having the same area. Since the area of $\triangle ABC$ is 1, the area of $\triangle DEF$ is 7.

6 Solutions: Harder Problems

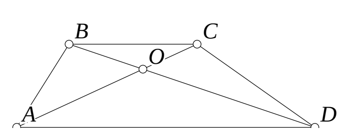
1. Consider the figure below where we have constructed the segments $A'C$ and AC:



The three triangles $\triangle ABC$, $\triangle A'BC$ and $\triangle A'B'C$ all have the same area. Since $AB = A'B$ the triangles $\triangle ABC$ and $\triangle A'BC$ have the same area since they have equal bases and the same height. Similarly, since $BC = B'C$ the triangles $\triangle A'BC$ and $\triangle A'B'C$ have equal bases and the same height.

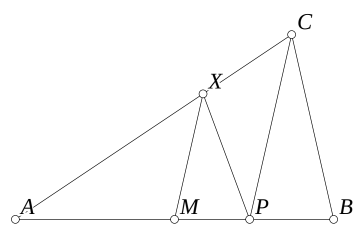
Thus the area of $\triangle ABC$ is half the area of $\triangle A'B'B$. We can make a similar argument about all of the outer triangles to show that each is double the area of one of the triangles formed by the bisection of the quadrilateral $ABCD$. Those four triangles: $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ and $\triangle DAB$ cover the original quadrilateral exactly twice so the sum of the outer triangles' areas is exactly twice the area of the quadrilateral $ABCD$. The area of $A'b'C'D'$ is the sum of the areas of the outer triangles and the area of $ABCD$, so it is 5 times as large as the area of $\triangle ABCD$.

2. See the figure below:



Since $BC \| AD$ we know that both B and C are the same distance from the line AD.l Triangles $\triangle ABD$ and $\triangle ACD$ have the same base and the same height and hence, the same area. If we subtract the area of $\triangle AOD$ from both of those equal areas we obtain the areas of triangles $\triangle AOB$ and $\triangle COD$ so those must also be the same.

3. See the figure below:



Assume the point P is closer to B than A so the midpoint M lies between A and P. We need to find a point X on the segment AC such that the area of $\triangle AXP$ is half the area of $\triangle ABC$.

We could drop perpendiculars from X and C to X' and C' on AB but since we're only interested in ratios of areas, we will have $XX' : CC' = AX : AC$. Using AX and AC, the point X must be found such that:

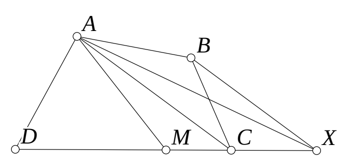
$$AP \cdot AX = \frac{1}{2} AC \cdot AB$$

and since $AM = AC/2$ we cam write:

$$AP \cdot AX = AM \cdot AB.$$

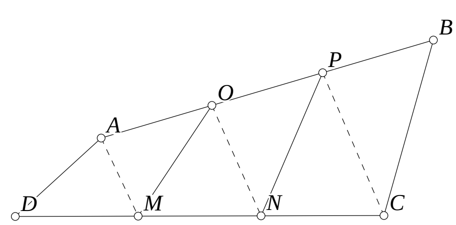
Thus we need to multiply the segment AC by AM/AP and that can be done by constructing the line PC and then constructing the line through M parallel to PC and letting X be the intersection of that line with AC.

4. See the figure below:



Through the point B draw a line parallel to AC that intersects the extension of the line DC at X. Since $BX \| AC$ we know that the triangles $\triangle ABC$ and $\triangle ACX$ have the same area. Thus for any point M between C and D, the area of $\triangle AMX$ is equal to the area of the quadrilateral $ABCM$. In particular, if M is the midpoint of DX it will split the area of $\triangle AXD$ into two equal-area triangles: $\triangle AMD$ and $\triangle AMX$. But the area of $\triangle AMX$ is the same as the area of the quadrilateral $ABCM$, so the line AM performs the desired division.

5. See the figure below:



We divide the quadrilateral into six triangles as shown in the figure: $\triangle DAM$, $\triangle AMO$, et cetera. First consider the sequence of triangles $\triangle DAM$, $\triangle MON$ and $\triangle NPC$. They have equal bases since M and N trisect DC so their areas are in the ratio of the altitudes from A, O and P. Since AB is a straight line, these altitudes are in a linear progression, so the area will increase by the same amount as we go from one to the next. If the area of $\triangle DAM$ is X, then the areas of those three triangles can be expressed as: $X, X+t, X+2t$ where t may be positive, negative or zero.

Now look at the triangles $\triangle AOM$, $\triangle ONP$ and $\triangle PCB$. Their altitudes from M, N and C also increase (or decrease) by a constant amount at each step, so if the area of $\triangle AOM$ is Y, the three areas will be $Y, Y+s, Y+2s$, again with s being positive, negative or zero.

The area of quadrilateral $MOPN$ is thus $X+Y+s+t$ and the area of the entire quadrilateral $ABCD$ is $3X+3Y+3s+3t$ which is three times the area of $MOPN$, wheich is what we wanted to show.

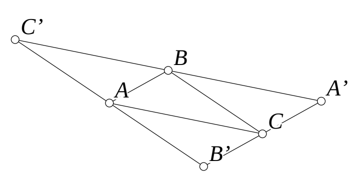
6. Imagine doing the same divisions as in the previous problem, but first in one direction and then in the other. The first division lines will divide the quadrilateral into three areas where the area of the middle one will be $1/3$ that of the entire one.

   The second set of dividing lines will divide the new edges of that middle piece into three equal lengths, and again, using the same argument, the area of the middle middle piece will be $1/3$ of the area of the middle piece, or $1/9$ the area of the original quadrilateral.

7. Suppose that there is an airport that receives 6 (or more) planes. We'll see what's wrong with 6 and the same argument will work for any larger number. If 6 planes land at airport A, say from airports B, C, D, E, F and G arranged in clockwise order around A, then for any triangle, say $\triangle ABC$ since AB and AC have to be smaller than BC (or the B and C planes would have gone to C and B, respectively) then since larger angles are opposite larger sides in any triangle, the angle at A is larger than the other two. Since the three angles add to $180°$ that means that the angle at A is larger than $60°$. The same thing can be said for every other triangle that includes A, so the total of the angles at A must be larger than $6 \cdot 60° = 360°$. This is more than a full circle so the situation is impossible.

8. Among all triples of points, find a particular set of points A, B and C so that $\triangle ABC$ has the largest area, which is 1 or less. If there is more than one set of such points, choose any particular one.

   Now construct a triangle $\triangle A'B'C'$ as in the figure below such that $AB \| A'B'$, $BC \| B'C'$ and $CA \| C'A'$.

   We claim that all the additional points must lie somewhere inside $\triangle A'B'C'$. If a point P is outside WLOG we can assume it is on the opposite side of $A'B'$ from A and B.

Consider the triangle $\triangle ABP$. Using base AB, its altitude is larger than that of $\triangle ABC$, so its area is larger than $\triangle ABC$, which is a contradiction.

9. No. Consider the convex hull of the endpoints of the segments. (The convex hull of a set is the smallest convex body that contains all the points in the set.) Since the points on the line segments lie between the endpoints and since the convex hull is convex, all the line segments will lie within the convex hull as well.

   The convex hull will be a polygon whose vertices are all endpoints of the line segments. Any one of these hull vertices cannot lie on the interior of any segment in the set, since any such segment would lie at least partly outside the convex hull.

10. If one of the points, say A, lies inside the triangle formed by points B, C and D, then the three triangles $\triangle ABC$, $\triangle ACD$ and $\triangle ADB$ meet at the point A and the three angles there add to $360°$. That means that at least one of those angles is at least $120°$, so there is a non-acute triangle.

    If no such point exists, then the quadrilateral (perhaps with renaming some points) $ABCD$ is convex. The four interior angles add to $360°$ so at least one interior angle of $ABCD$, say A, is $90°$ or more. That means that $\triangle ABD$ is non-acute.

11. First notice that no two rectangles can have the same n or same m. If (nk_1) and (n, k_2) where $k_2 > k_1$ are two rectangles in the collection, then the (n, k_2) rectangle completely contains the (n, k_1) rectangle.

    Assume that there is such a collection. Pick any particular rectangle (n, m) in the set. There are at most $n - 1$ rectangles with a smaller first coordinate and at most $m - 1$ rectangles with a smaller second coordinate. Since there are an infinite number of rectangles in the collection, all but at most $m + n - 2$ of them have larger first and second coordinates and thus there are an infinite number of rectangles that cover (n, m).

12. For part (a) in fact it is possible to find a rectangle of area 2 that contains the entire convex region. To see this, first find points A and B that are farthest apart in the region. Construct lines perpendicular through A and B perpendicular to AB. No other points of the region can lie on these lines since if another one did, say C on the line through A, then CB would be longer than AB.

    The entire convex region is thus contained between those two lines and let's arrange the figure so they are horizontal.

    Now, starting from the far left and far right, slide in two other lines parallel to AB until each one touches the convex region at L on the left and at R on the right. When that happens, we have a rectangular box that encloses the entire convex region.

    Note that the area of the box is twice the sum of the areas of the two triangles $\triangle ABL$ and $\triangle ABR$. Because of the convexity of the original figure, quadrilateral $ALBR$ is completely contained within it, so its area is less than or equal to 1. The are of the box is thus less than or equal to 2 and contains the entire convex region, so we are done.

    For part (b), note than any enclosing parallelogram must share two sides with the triangle. If not, by rotating the sides of the parallelogram to be closer to the sides of the triangle would reduce the area of the parallelogram. Thus the parallelogram shares two sides of the triangle, and thus must have an area of twice that of the triangle.

13. This is known as "Helly's Theorem." Here is a proof (basically from the page on Helly's Theorem in `cut-the-knot.org`).

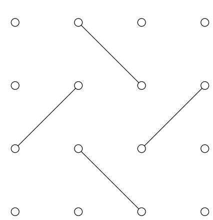
    First prove it for the case of four regions, R_1, R_2, R_3 and R_4. For every three regions, let p_i be a point that is in the three regions that do not include region R_i, so, for example, p_2 is a point in the intersection of R_1, R_3 and R_4, et cetera. There are two cases to consider:

    (a) One of the points, say p_1, lies in the interior of the triangle $\triangle p_2p_3p_4$. By the convexity of the regions, the entire triangle lies completely within regions R_2, R_3 and R_4, so p_1 also lies in all three regions, and that implies that p_1 lies in all four regions.

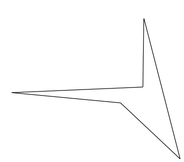
    (b) If the situation above does not occur, the points p_1, p_2, p_3 and p_4 form a convex quadrilateral since none of them lie within the triangle formed by the other three. By renaming, if necessary, assume that the order of the points around the quadrilateral is in numerical order of the indices. Then the diagonal p_1p_3 lies in both R_1 and R_3 and hence also all the points on that diagonal. The same can be said about the diagonal p_2p_4, but those lines intersect at a point p which must therefore be in all four regions.

    The proof above can be extended by induction to any number of sets R_i. It's true for 4 sets, so assume it's true for k sets and consider the situation with $k+1$ sets. Then let $G_i = R_i \bigcap R_{k+1}$ for $i < k+1$. Clearly the intersection of any three G_i is non-empty so by the induction hypothesis, the intersection of all of them is non-empty. But their intersection is the intersection of all the R_i ($k+1$ sets).

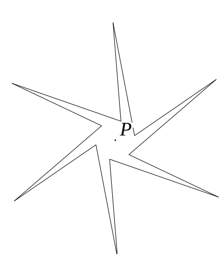
14. Not true. Here's a counterexample:



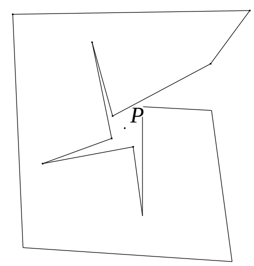
15. It is not true. See the figure below, for example:



16. See the figure below:

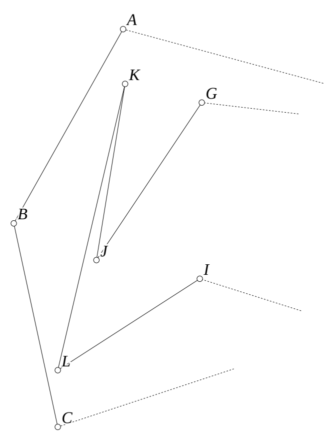


17. See the figure below:



18. Shrink the original rectangle to half its width and half its height (including the covering circles). The resulting figure will be completely covered by 100 circles of radius 1. Put four of these together to form a rectangle of the original size, but completely covered by 400 circles of radius 1.

19. (a) See the figure below:



The proof goes as follows. Find an angle $\angle ABC$ such that the interior of the polygon is on the side of the angle less than $180°$. Then there are two cases. Either the line segment AC lies completely within the polyon in which case we are done, and AC is the required diagonal, or some part of the polygon (shown as $GJKL$ in the figure) goes inside $\triangle ABC$.

Since there are only a finite number of vertices of the polygon interior to $\triangle ABC$; for each of those, construct a line perpendicular to the angle bisector of $\angle ABC$. Clearly, the line connecting B to the vertex with perpendicular closest to point B will lie completely within the polygon. If not, it had to cross another edge of the polygon, and one end of that edge would have a perpendicular to the angle bisector closer to B.

Note that we *cannot* use the vertex closest to B. In the figure, J is the point nearest B, but clearly segment JB crosses segment KL.

(b) In a non-convex 4-sided figure there is exactly one such diagonal.

In general, we know that any n-sided simple polygon can be completely triangulated by diagonals contained within the polygon, that at least $n - 3$ diagonals must exist. It is quite easy to construct a polygon that has exactly $n - 3$ diagonals and no more.

(c) Prove this by induction. A 3-sided figure must be a triangle, and so no additional diagonals are needed. Next assume that every polygon with $n \geq 3$ or fewer sides can be triangulated using only diagonals, and consider an arbitrary polygon with $n + 1$ sides. By part (a) above, we can find an interior diagonal for this polygon that will split it into two polygons, each of which has fewer than n sides. By the induction hypotheses, each of these can be triangulated using only interior diagonals, so the combination of those diagonals plus the original diagonal we used to split the $(n + 1)$-sided polygon will do the trick.

(d) Again we can prove this by induction. It's obviously true for a triangle, and if we split a larger polygon using an interior diagonal, label both ends of that diagonal with different colors, and then color both polygons into which the original polygon is subdivided, beginning with the two colors that you assigned to the ends of the diagonal you added.

20. Consider the following labeling of the squares of the 10×10 chessboard:

A	*B*	*C*	*D*	*A*	*B*	*C*	*D*	*A*	*B*
B	*C*	*D*	*A*	*B*	*C*	*D*	*A*	*B*	*C*
C	*D*	*A*	*B*	*C*	*D*	*A*	*B*	*C*	*D*
D	*A*	*B*	*C*	*D*	*A*	*B*	*C*	*D*	*A*
A	*B*	*C*	*D*	*A*	*B*	*C*	*D*	*A*	*B*
B	*C*	*D*	*A*	*B*	*C*	*D*	*A*	*B*	*C*
C	*D*	*A*	*B*	*C*	*D*	*A*	*B*	*C*	*D*
D	*A*	*B*	*C*	*D*	*A*	*B*	*C*	*D*	*A*
A	*B*	*C*	*D*	*A*	*B*	*C*	*D*	*A*	*B*
B	*C*	*D*	*A*	*B*	*C*	*D*	*A*	*B*	*C*

Notice that no matter where you place it on the board, a 1×4 domino will cover exactly one each of A, B, C and D. But if you count the numbers of each letter on the board, you will see that there are 25 A's, 26 B's, 25 C's and 24 D's. In order to have a valid covering, we need equal numbers of each so this problem is impossible to solve.